The Nullstellensatz degree of Tseitin formulas is the toggle width: cutwidth plus one on graphs of maximum degree three

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Abstract

Let G be a connected graph with an odd charge c, and let $T(G, c)$ be its Tseitin formula, written as polynomials clause by clause. We show that over every field of characteristic other than 2 the Nullstellensatz degree of $T(G, c)$ equals the *toggle width* of G: the least k such that the empty vertex set can be turned into $V(G)$ by adding or removing one vertex at a time, where toggling v at the set U costs $|\delta(U) \cup E(v)|$. When G has at least three vertices, the toggle width lies between $\mathrm{cw}(G) + 1$ and $\mathrm{cw}(G) + \lfloor \Delta(G)/2 \rfloor$, where cw is the cutwidth and Δ the maximum degree. So on graphs of maximum degree at most three the degree is exactly $\mathrm{cw}(G) + 1$. Over characteristic 2 it is $\Delta(G)$. The upper bound is a refutation with integer coefficients that sweeps the charge across a layout. The lower bound is a design whose values are signed sums over cuts. In the ± 1 encoding, with degree measured modulo $y_e^2 = 1$, the same arguments give exactly $\mathrm{cw}(G)$. A standalone script replays both bounds, and recomputes the degrees directly, on ten cubic graphs including the Petersen graph.

1 Results

Let $G = (V, E)$ be a connected graph and $c\colon V \to \{0, 1\}$ a charge with $\sum_v c(v)$ odd. The Tseitin formula $T(G, c)$ has a Boolean variable x_e for each edge and, for each vertex v, the constraint $\sum_{e \in E(v)} x_e \equiv c(v) \pmod 2$, where $E(v)$ is the set of edges at v. It is unsatisfiable [Tse68].

Encoding. For each vertex v and each assignment β of $E(v)$ that violates its constraint, the clause polynomial is $\chi_{v,\beta} = \prod_{e:\beta_e=1} x_e \prod_{e:\beta_e=0} (1 - x_e)$, of degree $|E(v)|$. A *Nullstellensatz refutation of degree d* over a field F is an identity

$$\sum_{v,\beta} q_{v,\beta}\, \chi_{v,\beta} + \sum_e r_e\, (x_e^2 - x_e) = 1$$

in $F[x]$ in which every product has degree at most d. We write NS_F for the least such d. This is the usual translation of the CNF formula, one polynomial per clause.

Widths. For $U \subseteq V$ let $\delta(U)$ be the set of edges with exactly one end in U.

- A *toggle path* is a sequence $\emptyset = U_0, U_1, \ldots, U_m = V$ in which each U_i arises from U_{i-1} by adding or removing one vertex v_i. The i-th step *costs* $|\delta(U_{i-1}) \cup E(v_i)|$.
- The *toggle width* $\mathrm{tgw}(G)$ is the least k such that some toggle path has every step costing at most k.

*Draft of 20 October 2026, prepared with Claude (Anthropic). A standalone script (`replay.py`) checks every certificate described here on ten small graphs. No human expert has reviewed this note.

- The *cutwidth* $\mathrm{cw}(G)$ is the least, over vertex orderings, of the largest $|\delta(\cdot)|$ of a prefix.

Theorem 1. *Let F be a field. If* $\operatorname{char} F = 2$ *then* $\mathrm{NS}_F(T(G,c)) = \Delta(G)$. *Otherwise* $\mathrm{NS}_F(T(G,c)) = \mathrm{tgw}(G)$, *whatever the charge.*

Theorem 2. *If G is connected with at least three vertices, then* $\mathrm{cw}(G) + 1 \le \mathrm{tgw}(G) \le \mathrm{cw}(G) + \lfloor \Delta(G)/2 \rfloor$.

Corollary 3. *If G is connected with at least three vertices and maximum degree at most three, and* $\operatorname{char} F \neq 2$, *then* $\mathrm{NS}_F(T(G,c)) = \mathrm{cw}(G) + 1$.

graph	n	$\lvert E\rvert$	treewidth	cw	NS_F, $\operatorname{char} F \neq 2$	NS_F, $\operatorname{char} F = 2$
K_4	4	6	3	4	5	3
triangular prism	6	9	3	4	5	3
$K_{3,3}$	6	9	3	5	6	3
cube Q_3	8	12	3	5	6	3
Wagner graph	8	12	4	5	6	3
Petersen graph	10	15	4	6	7	3

Table 1: Values given by Corollary 3. `replay.py` also obtains every entry of the last two columns by direct Gaussian elimination over $\mathbb{F}_3$ and $\mathbb{F}_2$. The degree is not a function of treewidth: K_4 and $K_{3,3}$ both have treewidth 3.

For bounded-degree expanders the cutwidth is linear in n, so these degrees are linear in n, in line with Grigoriev's linear lower bounds for Nullstellensatz refutations of Tseitin formulas [Gri98]. Theorem 1 gives the exact value for every graph. (The linear lower bounds of Buss, Grigoriev, Impagliazzo and Pitassi [BGIP01] are for the stronger polynomial calculus and are not implied.)

2 Designs as pseudo-distributions

Definition 4. A *pseudo-distribution of level d* over F assigns to each set $S \subseteq E$ with $|S| \le d$ a function $z_S \colon \{0,1\}^S \to F$ such that

- $z_\emptyset = 1$;
- $z_S(f) = z_{S\cup\{e\}}(f,0) + z_{S\cup\{e\}}(f,1)$ whenever $|S \cup \{e\}| \le d$ and $e \notin S$;
- $z_S(f) = 0$ whenever f violates the constraint of some vertex v with $E(v) \subseteq S$.

Lemma 5. $\mathrm{NS}_F(T(G,c)) > d$ *if and only if a pseudo-distribution of level d over F exists.*

Proof. By linear duality, no refutation of degree d exists if and only if some linear functional D on the polynomials of degree at most d satisfies $D(1) = 1$ and vanishes on every product $q\chi_{v,\beta}$ and $r(x_e^2 - x_e)$ of degree at most d.

From D to z. Given such a D, put $z_S(f) = D(\iota_{S,f})$, where $\iota_{S,f} = \prod_{e\in S}(x_e \text{ if } f_e = 1,\ 1 - x_e \text{ if } f_e = 0)$ has degree $|S|$.

- Consistency holds because $\iota_{S,f} = \iota_{S+e,(f,0)} + \iota_{S+e,(f,1)}$.
- $z_\emptyset = D(1) = 1$.
- If f violates v and $E(v) \subseteq S$, then $\iota_{S,f} = \chi_{v,f|E(v)} \cdot \iota_{S\setminus E(v),f}$ is a multiple of an axiom of degree $|S| \le d$, so $z_S(f) = 0$.

From z to D. Conversely, given z, define D on multilinear monomials by $D(x_W) = z_W(1_W)$, and on any polynomial of degree at most d by first reducing exponents with $x_e^2 = x_e$.

- $D(1) = z_\emptyset = 1$, and D vanishes on the multiples of $x_e^2 - x_e$.
- By consistency, $D(P) = \sum_{f\in\{0,1\}^S} z_S(f)P(f)$ for every multilinear P whose variables lie in S with $|S| \le d$.
- A multiple $x_T\chi_{v,\beta}$ of degree at most d reduces either to 0 or to such a P on $S = T\cup E(v)$. That P vanishes except at assignments restricting to the violating β, where z_S vanishes, so $D(x_T\chi_{v,\beta}) = 0$.
- Every multiple $q\chi_{v,\beta}$ is a combination of these. □

3 Upper bounds

Characteristic 2. Over $\mathbb{F}_2$ the multilinear polynomial $\sum_\beta \chi_{v,\beta}$, summed over the violating β, is the indicator of a violation at v. That indicator equals $\sum_{e\in E(v)} x_e + c(v)$. Summing over all vertices counts every x_e twice, so

$$\sum_v \sum_\beta \chi_{v,\beta} = \sum_v c(v) = 1,$$

a refutation of degree $\Delta(G)$.

Conversely, below degree $\Delta(G)$ the axioms of a vertex u of maximum degree cannot be used. Without u's constraint the formula is satisfiable (G is connected), and evaluating at a satisfying assignment turns any identity $\sum q\chi = 1$ that avoids u's axioms into $0 = 1$. This works over every field, so $\mathrm{NS}_F \ge \Delta(G)$ always.

The sweep refutation, over every field. Let (U_i) be a toggle path whose steps cost at most k.

- *Parities along the path.* Put $t_0 = 0$ and $t_i = t_{i-1} + c(v_i) \bmod 2$. Let $\Pi_{U,t}$ be the multilinear polynomial of the Boolean function $[\sum_{e\in\delta(U)} x_e \equiv t \pmod 2]$; its coefficients are integers.
- *One step.* Put $P_i = \Pi_{U_i,t_i} - \Pi_{U_{i-1},t_{i-1}}$. Its variables lie in $S_i = \delta(U_{i-1})\cup E(v_i)$, with $|S_i| \le k$.
- *P_i vanishes where v_i is satisfied.* Since $\delta(U_i) = \delta(U_{i-1})\triangle E(v_i)$, the two parities differ by $\sum_{e\in E(v_i)} x_e$. So P_i vanishes at every point of $\{0,1\}^{S_i}$ satisfying v_i's constraint.
- *P_i is a combination of axiom multiples.* Interpolating over $\{0,1\}^{S_i}$,

$$P_i = \sum_{f \text{ violating } v_i} P_i(f)\,\chi_{v_i,f|E(v_i)}\,\iota_{S_i\setminus E(v_i),f},$$

  an integer combination of multiples $x_T\chi_{v_i,\beta}$ with $T\cap E(v_i) = \emptyset$ and $|T| + |E(v_i)| \le |S_i| \le k$.
- *Telescoping.* Since $\delta(\emptyset) = \delta(V) = \emptyset$ and $t_m = \sum_v c(v) = 1$, we get $\sum_i P_i = \Pi_{V,1} - \Pi_{\emptyset,0} = 0 - 1$.

So $-\sum_i P_i = 1$ is a refutation of degree at most k with integer coefficients. It holds as a polynomial identity, without the Boolean axioms. Hence $\mathrm{NS}_F \le \mathrm{tgw}(G)$ for every field F.

4 The lower bound outside characteristic two

Let $\mathrm{char}\,F \ne 2$ and $k < \mathrm{tgw}(G)$. We build a pseudo-distribution of level k.

Reachable sets and signed cuts. Call U *reachable* if some toggle path from $\emptyset$ to U has every step costing at most k.

- *Reversible and complement-symmetric.* Toggling v costs the same from U and from $U\triangle\{v\}$, because $\delta(U) \cup E(v) = \delta(U\triangle\{v\}) \cup E(v)$. Complementing a path preserves its costs, since $\delta(V \setminus U) = \delta(U)$.
- *One representative per cut.* If U and $V \setminus U$ were both reachable, then V would be reachable through U: complement the path to $V \setminus U$ and reverse it. That contradicts $k < \mathrm{tgw}(G)$. Since G is connected, $\delta(U) = \delta(U')$ only for $U' \in \{U, V \setminus U\}$. So each cut γ has at most one reachable representative.
- *The sign function.* Put $h(\gamma) = (-1)^{c(U)}$ if $\gamma = \delta(U)$ with U reachable, where $c(U) = \sum_{u\in U} c(u)$, and $h(\gamma) = 0$ if γ has no reachable representative. In particular $h(\emptyset) = 1$.

Lemma 6. *Let $|S| \le k$, let $\gamma \subseteq S$ be a cut, and let w be a vertex with $E(w) \subseteq S$. Then $h(\gamma\triangle E(w)) = (-1)^{c(w)} h(\gamma)$.*

Proof. Write $\gamma = \delta(U)$. Then $\gamma\triangle E(w) = \delta(U\triangle\{w\})$, and toggling w at U costs $|\gamma\cup E(w)| \le |S| \le k$. So U is reachable exactly when $U\triangle\{w\}$ is; the same holds for their complements, and the charge changes by $c(w)$. □

The pseudo-distribution. For $|S| \le k$ and $f \in \{0,1\}^S$, let $\mathcal{C}(S)$ be the set of cuts contained in S and put

$$z_S(f) = 2^{-|S|} \sum_{\gamma\in\mathcal{C}(S)} h(\gamma)\,(-1)^{\sum_{e\in\gamma} f_e}.$$

- *Normalization.* $z_\emptyset = h(\emptyset) = 1$.
- *Consistency.* Summing $z_{S\cup\{e\}}(f, b)$ over b cancels every cut that contains e and doubles the others. The cuts in $S \cup \{e\}$ that avoid e are exactly $\mathcal{C}(S)$.
- *Vanishing.* Let f violate the constraint of v, with $E(v) \subseteq S$.
  - The map $\gamma \mapsto \gamma\triangle E(v)$ is a fixed-point-free involution of $\mathcal{C}(S)$.
  - By Lemma 6, it multiplies h by $(-1)^{c(v)}$. Since $\sum_{e\in E(v)} f_e \equiv c(v)+1$, it multiplies the character by $(-1)^{c(v)+1}$.
  - So paired terms cancel, and $z_S(f) = 0$.

By Lemma 5, $\mathrm{NS}_F > k$. With the sweep this proves Theorem 1.

5 Toggle width and cutwidth

Lemma 7 (monotone bound)**.** *Take an ordering whose prefix cuts are at most* $\mathrm{cw}(G)$*. When v is added to the prefix U and has d neighbours in U, the step costs*

$$|\delta(U)| + \deg v - d = |\delta(U \cup \{v\})| + d \le \mathrm{cw}(G) + \min(d, \deg v - d).$$

Hence $\mathrm{tgw}(G) \le \mathrm{cw}(G) + \lfloor \Delta(G)/2 \rfloor$.

Lemma 8 (peak elimination)**.** *Let G be connected with at least three vertices. If a toggle path has every step costing at most k, then some toggle path has every step costing at most k and every visited cut of size at most $k-1$.*

Proof. Call a visited set U with $|\delta(U)| = k$ a *peak*.

- *Peaks are interior.* The cost of a step bounds the cuts on both of its sides, so every visited cut is at most k. The endpoints have cut 0.

- *The neighbouring toggles.* Let a peak U be entered by toggling v and left by toggling w. Entering costs at most $|\delta(U)|$, which forces $E(v) \subseteq \delta(U)$; likewise $E(w) \subseteq \delta(U)$.
- *If* $v = w$, delete the detour.
- *Otherwise* replace U by $X = U \triangle \{v, w\}$, toggling w before v.
  - Then $|\delta(X)| = k - \deg v - \deg w + 2[vw \in E] \leq k - 1$. A cut of size k would need v and w adjacent and both of degree one, which connectivity and $n \geq 3$ exclude.
  - The two new steps cost $k - \deg v + [vw \in E]$ and $k - \deg w + [vw \in E]$, both at most k.

Each step removes a peak and creates none. □

Lemma 9 (Fomin and Thilikos [FT03], Lemma 1; after Bienstock and Seymour [BS91])**.** *Let ρ be a submodular function on the subsets of V with nonnegative integer values. Suppose some sequence $\emptyset = A_1, \ldots, A_r = V$ has $\rho(A_i) \leq k$ and $|A_{i+1} \setminus A_i| \leq 1$ for every i, with removals unrestricted. Then a monotone such sequence exists.*

Proof. Choose a sequence minimizing $\sum \rho(A_i)$, and subject to that $\sum(|A_i| + 1)$.

- *Unions do not help.* Replacing A_i by $A_{i-1} \cup A_i$ keeps the step conditions. If $\rho(A_{i-1} \cup A_i) < \rho(A_i)$, the result would be a valid sequence with smaller $\sum \rho$. So $\rho(A_{i-1} \cup A_i) \geq \rho(A_i)$.
- *Intersections do not hurt.* By submodularity, $\rho(A_{i-1} \cap A_i) \leq \rho(A_{i-1})$.
- *So the sequence is monotone.* Replacing A_{i-1} by $A_{i-1} \cap A_i$ keeps the step conditions and does not raise $\sum \rho$. Minimality then forces $A_{i-1} \subseteq A_i$. □

Proof of Theorem 2. Take a toggle path of cost $k = \mathrm{tgw}(G)$.

- Lemma 8 turns it into a path whose cuts are all at most $k - 1$.
- Lemma 9, applied to $\rho = |\delta(\cdot)|$, turns that into an ordering of cutwidth at most $k - 1$.

So $\mathrm{cw}(G) \leq \mathrm{tgw}(G) - 1$. The monotone bound gives the other inequality. □

6 Remarks

- **The ± 1 encoding.** Let $\mathrm{char}\, F \neq 2$. Put $y_e = 1 - 2x_e$, $y_A = \prod_{e \in A} y_e$ and $s_v = (-1)^{c(v)}$, and write the constraint of v as the single binomial $y_{E(v)} - s_v$, with the axioms $y_e^2 - 1$.
  - *Degree counted before reduction.* As polynomials in x, $y_{E(v)} - s_v = -2s_v \sum_\beta \chi_{v,\beta}$ and $y_e^2 - 1 = 4(x_e^2 - x_e)$. So every refutation in this encoding becomes a clause-encoded refutation of no larger degree, and its degree is therefore at least $\mathrm{tgw}(G)$. Modulo $y_e^2 = 1$ the product $y_{\delta(U)}(y_{E(v)} - s_v)$ is $y_{\delta(U \triangle \{v\})} - s_v y_{\delta(U)}$, so these products, taken along an ordering of cutwidth $\mathrm{cw}(G)$, telescope to a refutation of degree at most $\mathrm{cw}(G) + \Delta(G)$.
  - *Degree counted after reduction modulo $y_e^2 = 1$.* The degree is then exactly $\mathrm{cw}(G)$, for every connected G. A product $y_A(y_{E(v)} - s_v)$ reduces to $y_{A \triangle E(v)} - s_v y_A$, of degree $\max(|A|, |A \triangle E(v)|)$, so the products above give degree $\mathrm{cw}(G)$. These two-term relations pair up the monomials, so the reduced multiples of one binomial of degree at most d span exactly the relations with $|A|, |A \triangle E(v)| \leq d$. For $d < \mathrm{cw}(G)$, call U reachable if a toggle path reaches it with every visited cut at most d. If U and $V \setminus U$ were both reachable, so would be V, as in Section 4, and Lemma 9 would give an ordering of cutwidth at most d. So we may put $L(y_{\delta(U)}) = (-1)^{c(U)}$ for reachable U and $L = 0$ on every other monomial. Then $L(1) = 1$, and as in Lemma 6 L satisfies every relation of degree at most d.
  - On the ten graphs of Table 1 the replay finds degree $\mathrm{cw}(G)$ after reduction and $\mathrm{tgw}(G) + 1$ before.

- **Sherali–Adams.** Requiring the pseudo-distributions to be nonnegative gives Sherali–Adams. Let tw be the treewidth of the variable graph, here the line graph of G. Then no nonnegative pseudo-distribution of level $\mathrm{tw}+1$ exists: gluing the local distributions along a tree decomposition would give a satisfying distribution [WJ04]. For the 10-vertex cubic graph with edges $01, 02, 03, 12, 14, 25, 36, 37, 46, 48, 57, 59, 68, 79, 89$, the line graph has treewidth 4 while $\mathrm{cw} = 5$. So Nullstellensatz needs degree 6 there, while Sherali–Adams refutes at degree 5.
- **Tseitin modulo p.** Let each edge carry a $\mathbb{Z}_p$-valued variable, orient the edges, and require at each vertex that the outflow minus the inflow equal $c(v)$, with $\sum_v c(v) \neq 0$. Define pseudo-distributions as above. Over any field of characteristic other than p they exist exactly below level $\mathrm{tgw}(G)$: the same proof works with characters of order p in place of signs and $\mathbb{Z}_p$-labellings u of the vertices in place of vertex sets. A labelling is reachable if it can be reached from 0 by changing one label at a time, a change at v costing $|\operatorname{supp} \delta u \cup E(v)|$, where δu is the induced labelling of the edges. If u and $u + a$ were both reachable for a nonzero constant a, then so would be the constant $-a$, and the level sets of the value $-a$ along the way would form a toggle path from $\emptyset$ to V of no larger cost. A p-th root of unity may be adjoined, since the conditions are linear with integer coefficients. Over a field of characteristic p the threshold is $\Delta(G)$.
- **Layout parameters.** For graphs of maximum degree three, cutwidth equals the search number (Makedon and Sudborough [MS89]), so Corollary 3 can be read in those terms.
- **Resolution.** Galesi, Talebanfard and Torán characterize the resolution width, variable space and depth of Tseitin formulas exactly by a cops-and-robber game, a variant of the Seymour–Thomas game for treewidth [GTT20].

Novelty. After a targeted search we did not find Theorem 1, Corollary 3 or the ± 1 statement above stated. The methods, sweeping refutations, character-sum designs and monotonicity of submodular games, are standard. We would be glad to learn of prior statements.

7 The replay script

`python3 replay.py` needs only the Python standard library. It runs on K_4, the prism, $K_{3,3}$, all five connected cubic graphs on eight vertices, the 10-vertex graph above, and the Petersen graph, each with charge 1 at one vertex. For each graph it checks:

1. the cutwidth, the toggle width and its monotone version;
2. the sweep refutation, as a polynomial identity over $\mathbb{Z}$, using products of degree at most $\mathrm{tgw}(G)$;
3. the pseudo-distribution of level $\mathrm{tgw}(G) - 1$, in its dual form as a design, against every axiom multiple of that degree, over $\mathbb{F}_3$, $\mathbb{F}_5$ and (up to twelve edges) $\mathbb{Q}$;
4. the characteristic-2 refutation;
5. the least degree at which 1 lies in the span of the axiom multiples, by Gaussian elimination over $\mathbb{F}_2$ and $\mathbb{F}_3$;
6. peak elimination and uncrossing on random toggle paths;
7. the least degree of the ± 1 encoding, before and after reduction, by signed union-find on the two-term relations;
8. the treewidth of G and of its line graph.

Controls must fail, and do:

- an even charge admits no refutation, in either encoding;
- the design breaks down at level $\mathrm{tgw}(G)$;
- the sweep with one step removed does not sum to 1.

The run takes about half a minute.

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