Counting 2-colourings of 3-regular hypergraphs with Pfaffian incidence graphs

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Abstract

Counting the proper 2-colourings of a 3-regular 3-uniform hypergraph is #P-hard (Fan and Cai), and polynomial when its incidence graph is planar (Valiant). We study the class in between where the incidence graph is Pfaffian. It contains every hypergraph that cannot be 2-coloured: by Seymour and by Robertson, Seymour and Thomas, these are exactly the ones whose incidence matrix is sign-nonsingular. Our results: (A) for every connected non-2-colourable such hypergraph H and every edge L, the number of 2-colourings of $H - L$ is computable in polynomial time, as a product over braces of determinants assembled along trisum trees by a three-term "trisum law"; (B) with signed incidences, counting stays polynomial as long as the signs differ from a Pfaffian signing by a twist that does not branch, because a junction with one untwisted side multiplies the matchgate identity by $\alpha^2 + \alpha\beta + \beta^2$; (C) once twists branch, counting is #P-hard on Pfaffian graphs; (D) it remains #P-hard without any signs; and (E) it remains #P-hard when the incidence graph is a Pfaffian brace. The hardness proofs build crossings inside Pfaffian graphs: junctions of twisted pieces cross softly, and ladders of plain cube junctions cross exactly after interpolation. Along the way we find that every plain planar piece is twisted, and that the Fano plane is the only non-2-colourable 3-regular 3-uniform hypergraph whose incidence graph is a brace.

1 Results

A *cubic triple system* H has n points and n lines; every line has three distinct points and every point lies on three lines. Its incidence graph $B = B(H)$ is a cubic bipartite graph. A 2-colouring of the points is *proper* if no line is monochromatic, and $\#\mathrm{col}(H)$ counts them. In the language of counting complexity this is $\mathrm{Holant}([0, 1, 1, 0] \mid =_3)$, or not-all-equal 3-SAT with positive literals in which every variable occurs three times.

Fan and Cai [5] proved #col #P-hard; Valiant's holographic algorithm [13] computes it when B is planar. We consider incidence graphs that are *Pfaffian*, the class on which perfect matchings can be counted by a single Pfaffian [6, 7, 10, 9]. We also use *signed* systems: a set σ of incidences, at which a line sees the complement of the point's colour.

Theorem 1 (counting on Seymour's bound)**.** *Let H be a connected cubic triple system that is not 2-colourable, and L one of its lines. Then $\#\mathrm{col}(H - L)$ can be computed in time polynomial in n. More precisely, $\#\mathrm{col}(H - L) = 2\prod_K a(K)$, the product over the braces K of $B(H)$, where $a(K) = 1$ for the two degenerate braces, $a(K) = 5$ for the Heawood graph, $a(K) = |\det|$ of a line–face matrix*

*Draft of 31 October 2026, prepared with Claude (Anthropic). The results come from proofs with executable checks in the public repository https://github.com/jscul/knotlab; Section 9 lists them, a replay driver reruns every cited check, and a separately written script recomputes the headline numbers. No human expert has reviewed this note.

for a planar brace, and $a(K)$ is given by the trisum law of Theorem 8 with two determinants per planar leaf otherwise. The same holds for every connected cubic signed system that is not colourable.

Theorem 2 (unbranched twists)**.** *Let T be a cubic Pfaffian brace with a trisum tree, κ a Pfaffian signing and σ any signing. If no square of the tree has all three of its branches twisted (Definition 7), then $\#\mathrm{col}(T, \sigma)$ is computable in polynomial time. The same holds for every connected cubic bipartite Pfaffian graph through its tight cuts.*

Theorem 3 (hardness under signings)**.** *Counting $\#\mathrm{col}(B, \sigma)$ for cubic bipartite Pfaffian graphs B and signings σ is $\#\mathrm{P}$-hard under polynomial-time Turing reductions. In the instances used, every junction square has three twisted branches.*

Theorem 4 (no signs)**.** *Counting $\#\mathrm{col}(H)$ for cubic triple systems whose incidence graphs are Pfaffian is $\#\mathrm{P}$-hard under polynomial-time Turing reductions.*

Theorem 5 (braces)**.** *Counting $\#\mathrm{col}(H)$ for cubic triple systems whose incidence graphs are Pfaffian braces is $\#\mathrm{P}$-hard under polynomial-time Turing reductions.*

Theorems 1 and 2 describe where the Pfaffian structure helps: where the signs agree with a Pfaffian signing, counting is a determinant per planar piece, and twists that run along a path of the trisum tree do not change that. Theorems 3–5 show that nothing more survives. Deciding 2-colourability for these systems is polynomial (Robertson, Seymour and Thomas [10], McCuaig [9]); counting is not.

Two structural facts appear on the way (Section 6): every plain planar piece is twisted, so in a plain trisum brace all twists branch; and the Heawood graph is the only cubic brace whose plain system cannot be 2-coloured, that is, the Fano plane is the only non-2-colourable cubic triple system whose incidence graph is a brace. Among plain Pfaffian braces the polynomial cases we know are therefore planarity and the Fano plane (Corollary 21).

Provenance. The source of hardness is Fan and Cai's theorem (we also give an independent reduction from not-all-equal 3-SAT, which Creignou and Hermann [2] showed #P-complete). Planar tractability is Valiant's; the structure of Pfaffian braces is due to Robertson, Seymour and Thomas and to McCuaig; Lemma 6 is a signed form of Seymour's linear-algebra theorem [11]; Berman's determinant [1] is used for planar pieces. Bounded literature searches found no statement of Theorems 1–5; folklore risk is discussed per result in Section 9.

2 Preliminaries

Signed systems. A *signed system* is a cubic bipartite graph B (points and lines) with a set $\sigma \subseteq E(B)$ of negated incidences. A colouring c of the points is proper if every line ℓ sees the values $c(p) \oplus [p\ell \in \sigma]$ not all equal. Complementing a point's colour, or all three values a line sees, is a bijection between proper colourings for σ and for $\sigma \triangle \delta(v)$; so $\#\mathrm{col}(B, \sigma)$ depends only on the class of σ modulo cuts.

Pfaffian signings. A signing σ is *Pfaffian* if all perfect matchings of B contribute with the same sign to the determinant of the signed biadjacency matrix. A cycle C is *nice* if $B - V(C)$ has a perfect matching; a signing is Pfaffian if and only if every nice cycle carries $|C|/2 - 1$ negations modulo 2, and any two Pfaffian signings of a connected matching-covered bipartite graph differ by a cut [7, 8]. For planar graphs Kasteleyn's face condition gives one [6].

Lemma 6 (dictionary)**.** *A connected cubic signed system is not colourable if and only if its signing is Pfaffian. If it is not colourable, deleting any line makes it colourable.*

For plain systems this is the equivalence of non-2-colourability with sign-nonsingularity of the incidence matrix (Seymour [11]; see [10, Sect. 1]); the signed form is proved in the repository (`THREEFOLD-AMPLITUDE-20261030-001C`, Lemma 4). So the non-2-colourable cubic triple systems are those whose incidence graph is Pfaffian with the empty signing; we call them *three-fold spaces*.

Braces and trisums. A *brace* is a connected bipartite graph in which every matching of two edges extends to a perfect matching; the tight cut decomposition [4, 8] splits a matching-covered bipartite graph into braces. By Robertson, Seymour and Thomas [10] and McCuaig [9], a Pfaffian brace is planar, the Heawood graph, or a *trisum* of smaller Pfaffian braces: three braces sharing a 4-cycle, the *square*, glued along it with its four edges deleted. For a cubic Pfaffian brace T that is neither planar nor the Heawood graph, splitting repeatedly along squares gives a *trisum tree*: its leaves are planar braces, its squares are disjoint, each square is a face of the three leaves containing it, and each square has three *branches*, the components of the tree minus the square (`SEYMOUR-COUNT-20261030-001B`, Lemma 1). A square is written (c_0, c_1, c_2, c_3) with c_0, c_2 points and c_1, c_3 lines.

The Hadamard form. With $H = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ on every edge, a point becomes $2[1,0,1,0]$, a line $\frac{1}{4}[3,0,-1,0]$ and a negated edge $\mathrm{diag}(1,-1)$, so

$$\#\mathrm{col}(B,\sigma) = \left(\tfrac{3}{2}\right)^n \sum_{\chi} \left(-\tfrac{1}{3}\right)^{|\chi|/2} (-1)^{|\sigma\cap\chi|}, \tag{1}$$

the sum over even subgraphs χ of B (disjoint unions of cycles). Both constraint functions are matchgate signatures; this is Valiant's planar algorithm. A 4-ary even function f with legs in the cyclic order of a square is a matchgate signature if and only if $\mathrm{MGI}(f) = f_{0000}f_{1111} - f_{1100}f_{0011} + f_{1010}f_{0101} - f_{1001}f_{0110} = 0$.

Definition 7 (twists)**.** Let κ be a Pfaffian signing of B. The *twist* of a signing σ is $\sigma\triangle\kappa$ modulo cuts. In a trisum tree, the branch b of a square is *twisted* if $\sigma\triangle\kappa$, restricted to the edges of the leaves in b (square edges excluded), is not a cut of the graph they form.

A branch is untwisted if and only if some sign pattern on its square completes σ to a Pfaffian signing of the branch, and a branch containing a twisted branch is twisted (`TWIST-COUNT-20261030-001A`, Lemma 3).

3 Counting on Seymour's bound

Let P be a *piece*: a connected cubic signed system containing a square, with a Pfaffian signing in which the square carries an odd number of negations. Exactly one of c_1, c_3 then sees c_0 and c_2 equal. In a proper colouring of $P - c_1$, the line c_3 sees c_0 and c_2 different, and its third point agrees (as seen by c_3) with exactly one of them. The *leaning split* (α, β) counts half the colourings of $P - c_1$ in which it agrees with c_0, respectively c_2; $\rho = \alpha + \beta$ is the amplitude $\#\mathrm{col}(P - L)/2$, the same for every line L.

Theorem 8 (the trisum law)**.** *Three pieces in a common gauge, glued along their square, have amplitude*

$$A(T) = \rho_1\rho_2\rho_3 - \alpha_1\alpha_2\alpha_3 - \beta_1\beta_2\beta_3,$$

the number of choices of leanings, weighted by the splits, that are not all equal. Seen from c_3 the split is (β, α). A piece glued at a second square D to two pieces Q, R acts on the colourings of its own lines only through the weights $W_\alpha = \rho_Q\rho_R - \alpha_Q\alpha_R$ and $W_\beta = \rho_Q\rho_R - \beta_Q\beta_R$.

Proof outline. In a common gauge (every piece carries the same odd pattern on the square, Lemma 1 of `SEYMOUR-COUNT-20261030-001A`), fix the colours of c_0, c_2. In each piece the square line that sees them equal must be improper inside the piece, so its third point takes that value; the other square line is then proper and its third point leans to c_0 or c_2, with weights α_i or β_i. In the trisum, the square lines receive one point from each piece and must see not-all-equal values; counting gives the law (`SEYMOUR-COUNT-20261030-001A`, Theorem 3 and Propositions 4–5). □

Proposition 9 (a leaf with its children)**.** *For a planar leaf ℓ of a trisum tree, at a square S, let M be the matrix with rows the lines of ℓ other than c_1 and columns the faces not at c_1, with entry 1 for incidence, except that the rows d_1, d_3 of every other square D of ℓ carry W_α, W_β on D's two side faces (in opposite orders) and 1 on D's face. Restricting the row of c_3 to its side face through c_0, respectively c_2, gives matrices M_α, M_β, and the split of the branch at S is $(|\det M_\alpha|, |\det M_\beta|)$.*

Proof outline. For a planar brace, $|\det|$ of the line–face matrix counts Tutte matchings with outer triangle at the omitted line, all with one sign (Berman [1]); weights keep the sign coherence, and Theorem 8 identifies the weighted count with the split (`SEYMOUR-COUNT-20261030-001B`, Proposition 3 and Lemma 4). □

Proof of Theorem 1. By Lemma 6 and the tight cut decomposition, $\#\mathrm{col}(H - L)/2$ is a product of brace amplitudes, independent of L (`THREEFOLD-AMPLITUDE-20261030-001C`, Corollary 5). The degenerate braces contribute 1 and the Heawood graph 5. A planar brace contributes the determinant of its line–face matrix. Any other brace has a trisum tree (Section 2); its amplitude is the trisum law at one square, with each branch's split given by Proposition 9 from the splits one level down. That is two determinants per leaf, and no signing is needed at any step (`SEYMOUR-COUNT-20261030-001B`, Theorem 5). □

Remark 10. The checks compare the algorithm with direct counts on 67 braces, on three-fold spaces of 22 to 118 points (for example amplitudes 125 for a 22-point space and 3906250000 for a 90-point one, an amplitude being $\#\mathrm{col}(H - L)/2$), and on trisums of omnitruncated polyhedra up to 480 vertices; trisums of three omnitruncated 9×9 grids (1720 vertices) are counted in seconds, where variable elimination would need tables of width 30.

4 Twists

Let T be a cubic Pfaffian brace with a trisum tree and σ a signing. For a square S and a branch b, the *branch tensor* $F_{S,b}(z_0, z_1, z_2, z_3)$ counts the colourings of the branch's own points that make its own lines proper, where z_0, z_2 are the colours of c_0, c_2 and z_1, z_3 those of the neighbours of c_1, c_3 in b. Seen from a third branch, the *junction* of two closed branches Q, R is

$$W(x_0, z_1, x_2, z_3) = \sum F_Q(x_0, q_1, x_2, q_3) F_R(x_0, r_1, x_2, r_3)\ \mathrm{NAE}(z_1, q_1, r_1)\ \mathrm{NAE}(z_3, q_3, r_3),$$

with incidence signs complementing the corresponding arguments. In the Hadamard basis, branch tensors are even, and a single planar leaf is a matchgate signature under any signing.

Theorem 11 (the junction is a similitude)**.** *If Q is untwisted with split (α, β), then for every R*

$$\mathrm{MGI}(\hat{W}) = \pm(\alpha^2 + \alpha\beta + \beta^2)\ \mathrm{MGI}(\hat{F}_R),$$

the sign given by four incidence signs at the junction. In particular W is a matchgate signature if and only if F_R is.

Proof outline. An untwisted branch has the *split form*: a matchgate signature determined by (α, β), an odd pattern on the square and the leg signs (`TWIST-COUNT-20261030-001A`, Lemma 3). The identity is then a polynomial identity in α, β and the eight even entries of $\hat{F}_R$, verified symbolically for all eight odd patterns and all four choices of leg signs (ibid., Theorem 4). □

The multiplier is $|\alpha + \beta e^{i\pi/3}|^2$, a norm in the Eisenstein integers; it is positive, so an untwisted side never moves a junction off the matchgate class.

Proof of Theorem 2. If no square has three twisted branches, some leaf r sees at most one twisted branch among the two closed branches of every square (`TWIST-COUNT-20261030-001A`, Lemma 5). Rooted at r, every junction has an untwisted closed branch, so by Theorem 11 and induction every junction is a matchgate signature, and each leaf is a planar matchgate computation (FKT). Tight cuts contribute three-edge gadgets, which are always matchgates (ibid., Theorem 6). □

When twists branch the mechanism stops: junctions of two twisted pages leave the matchgate class. They still cannot cross wires sharply.

Proposition 12 (soft crossings)**.** *(i) With nonnegative branch tensors no junction is an exact crossover $[x_0 = x_2 \oplus a][z_1 = z_3 \oplus b]$: the square's lines forbid one pattern only, so on any slice confined to one parity class of (z_1, z_3) the junction pins z_1 (`TWIST-COUNT-20261030-001B`, Theorem 2). (ii) Junctions of two twisted truncated-octahedron pages include 256 that are* crossover-type *(zero turning entries in the Hadamard basis) with all four straight entries nonzero, for example* $(40000, -1112, 3336, -192)$ *(ibid., Proposition 3). We call such a junction a* soft crossing*.*

5 Hardness under signings

A *dead cap* is the cube minus an edge, attached to an outside line and an outside point: its count is 3 whatever they see, so a line with one capped leg acts on its other two legs by $3M$, $M = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$. A *$k$-snake* is k soft crossings in series, the wires joined between junctions by capped lines and points.

Lemma 13 (snakes and the lattice condition)**.** *The k-snake built from the soft crossing $\sigma_1 = (40000, -1112, 3336, -192)$ is crossover-type with straight entries $\sigma_1(s)\Lambda(s)^{k-1}$, where*

$$\Lambda = (810000,\ 7506,\ -22518,\ -432) = (2^4 3^4 5^4,\ 2 \cdot 3^3 \cdot 139,\ -2 \cdot 3^4 \cdot 139,\ -2^4 3^3).$$

The valuations at 5, 139 and 3 determine any count vector $n \in \mathbb{N}^4$ of given total from $\prod_s \Lambda(s)^{n_s}$, so the products $\prod_s \Lambda(s)^{2n_s}$ are distinct.

Proof. The multiplier per step is $\frac{9}{4}\mu(y_A)\mu(y_B)\sigma_1(s)$ with $\mu = (3, -1)$ the eigenvalues of M; the valuations are $v_5 = (4, 0, 0, 0)$, $v_{139} = (0, 1, 1, 0)$ and $v_3 = (4, 3, 4, 3)$, which give n_{00}, then $n_{01} + n_{10}$ and n_{10} (`SOFT-CROSS-20261031-001A`, Lemmas 2–3). □

Theorem 14 (universality inside Pfaffian graphs)**.** *Let G be a cubic signed system drawn in the plane with m crossings. Replace every edge by a capped path with k_E pairs and every crossing by a k_S-snake. The resulting instances are Pfaffian, and $\#\mathrm{col}(G)$ is computed in polynomial time from their counts for k_E in an interval of length $|E(G)| + 1$ and $k_S = 1, 3, \ldots, 2N - 1$, $N = \binom{m+3}{3}$.*

Proof outline. The host (everything outside the pages) is planar and every junction square is a face of it and of its pages, so Kasteleyn signings of all pieces, switched to a common odd pattern on each square, glue to a non-colourable, hence Pfaffian, signing (Theorem 8's gauge lemma and Lemma 6). Expanding long edges into $9^k P_+ + (-3)^k P_-$ and snakes into their straight states gives a sum over two families of distinct nodes (Lemma 13); a two-parameter Vandermonde system determines every coefficient, and replacing each snake by the compensated crossover $[z_1 = z_3]\, w(z_0, z_2)$ with $w = \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix}$ gives $9^m \#\mathrm{col}(G)$ (`SOFT-CROSS-20261031-001A`, Theorem 5). □

Proof of Theorem 3. Counting the colourings of plain cubic triple systems is #P-hard: this is Fan and Cai's theorem for Holant$([0, 1, 1, 0] \mid =_3)$, and the repository also derives it from positive not-all-equal 3-SAT [2] by joining the copies of a variable with capped paths, which act by $9^k(P_+ + (-\frac{1}{3})^k P_-)$ and interpolate to equalities (`SOFT-CROSS-20261031-001B`, Theorem 3). Theorem 14 reduces these to Pfaffian instances. Both pages of every junction are twisted, a cut restricts to a cut, and the host side of a junction in a snake contains other junctions' pages, so all three branches are twisted (ibid., Theorem 4). □

6 No signs

Lemma 15 (signs are path lengths)**.** *Replace each of the L negated incidences (p, ℓ) of a signed system by an unsigned capped path with k pairs, acting by $9^k(P_+ + tP_-)$ with $t = (-\frac{1}{3})^k$. The count is $9^{kL} q(t)$ for a polynomial q of degree at most L, and $q(-1)$ is the signed count while $q(1)$ is the count with the signs dropped.*

Proof. At $t = -1$ the path acts by $P_+ - P_- = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, the negation; the nodes $(-\frac{1}{3})^k$, $k = 1, \ldots, L+1$, are distinct (`PLAIN-COUNT-20261031-001A`, Lemma 1). □

Proof of Theorem 4. Capped paths drawn inside the planar pieces keep them planar, so the certificate of Theorem 14 survives. Switching inside the pages leaves six negations per junction. Lemma 15 then removes all signs from the instances of Theorem 3 (`PLAIN-COUNT-20261031-001A`, Theorem 4). □

Proposition 16 (plain pages are twisted)**.** *(i) Every 2-connected planar cubic bipartite graph has at least six square faces, and at most five faces meet a given square face. (ii) Hence every* plain page *(such a graph with a junction square face, no signs) is twisted. (iii) In a plain cubic Pfaffian brace that is a trisum, every square has three twisted branches, so Theorem 2 applies to none.*

Proof. (i) $\sum_f (6 - |f|) = 12$ by Euler's formula, and only squares contribute positively; a face meeting a square S at a vertex contains an edge of S. (ii) A square face Q disjoint from S is a cycle of the page's own edges, a Kasteleyn signing has $|Q|/2 - 1 = 1$ negation on it, and the plain signing has none, so they differ by a non-cut. (iii) Every branch contains a leaf of the tree, a plain page, and twistedness is inherited (`PLAIN-COUNT-20261031-001B`, Theorem 2 and Corollary 3). □

Proposition 17 (the Fano plane among braces)**.** *In a plain non-colourable system every nice cycle has length* 2 *modulo* 4*. Consequently the Heawood graph is the only cubic brace whose plain system is not 2-colourable: the Fano plane is the only non-2-colourable cubic triple system whose incidence graph is a brace.*

Proof. The empty signing is Pfaffian (Lemma 6), so a nice cycle C needs $|C|/2 - 1 \equiv 0$. In a brace every 4-cycle is nice, so a plain non-colourable brace has no 4-cycle. Planar cubic bipartite graphs have square faces (Proposition 16), and a trisum contains a square face of a leaf away from its junction; by Robertson, Seymour and Thomas only the Heawood graph remains (`PLAIN-COUNT-20261031-001B`, Corollary 5). □

This is probably implied by the description of minimal non-2-colourable square hypergraphs as splices of Fano planes [10]; the argument above is short and direct.

7 Braces

The instances so far contain 2-edge cuts (caps and capped paths), so they are not braces, and a single junction of plain pages is never a crossing.

Ladder sites. Replace a crossing of edges e and f by k junction squares, each glued by trisum to two plain cubes. Wire A runs from e's point end through l_s, then $c_0 \to c_2$ of each junction; wire B runs from f's line end through π_s, then $c_1 \to c_3$ of each junction. The start rung $l_s\pi_s$ and a rung $\lambda_j\pi_j$ between consecutive junctions close every leg; there are no caps.

Lemma 18 (the site is a matrix power)**.** *As a map from (colour of e's point end, colour of π_s) to (colour seen by e's line end, colour of f's point end) the site acts by $(NW)^k$, where W is the junction and $N[(z_2, z_3), (a, b)] = \mathrm{NAE}(z_2, a, b)[z_3 = b]$ the rung. Restoring e and f is the identity, the term $k = 0$.*

Lemma 19 (the cube junction)**.** *For two plain cubes,*

$$NW = \begin{pmatrix} 2 & 3 & 2 & 5 \\ 8 & 4 & 8 & 9 \\ 9 & 8 & 4 & 8 \\ 5 & 2 & 3 & 2 \end{pmatrix}, \qquad \det(xI - NW) = (x^2 - 19x - 1)(x^2 + 7x + 13).$$

The roots are $\varepsilon, -1/\varepsilon$ with $\varepsilon = (19 + \sqrt{365})/2$ a unit of norm -1, and $\pi, \bar{\pi}$ with $\pi = (-7 + i\sqrt{3})/2$ an Eisenstein prime of norm 13. For count vectors $n \neq n'$ of equal total, $\prod \lambda_i^{2n_i} \neq \prod \lambda_i^{2n'_i}$.

Proof. If two such products agree, $\varepsilon^{2(a-a')}$ (in $\mathbb{Q}(\sqrt{365})$) equals $\pi^{2(b'-b)}\bar{\pi}^{2(c'-c)}$ (in $\mathbb{Q}(\sqrt{-3})$), so both are rational; $\pi/\bar{\pi}$ is not a root of unity, so $b' - b = c' - c$, and a rational power of the unit ε is trivial (`PLAIN-BRACE-20261031-001A`, Lemmas 2–3). □

Consequently, with $Z(k)$ the count when a single site has k junctions, the count with the crossing restored is $\sum_{j=0}^{3} f_j Z(2j+1)$, where $f = (-\frac{53451621}{2804438}, \frac{521319}{200317}, -\frac{23524}{200317}, \frac{853}{2804438})$ is the unique cubic with $xf(x^2) \equiv 1$ modulo the characteristic polynomial; for m sites sharing an odd k, $\binom{m+3}{3}$ counts suffice. Only odd k gives a planar host. For $K_{3,3}$ drawn with one crossing, the plain instances with $k = 1, 3, 5, 7$ have 20, 48, 76 and 104 vertices and counts 80, 31500, 11459396 and 4159886188, and the formula returns $\#\mathrm{col}(K_{3,3}) = 6$.

Lemma 20. *(i) A connected cubic bipartite graph on at least six vertices that is cyclically 4-edge-connected is a brace. (ii) A site with k junctions is the ladder $P_{3k} \times K_2$ with its ports at the corners. (iii) Let a connected cubic bipartite graph G be drawn with its vertices in distinct faces of a cyclically 4-edge-connected planar cubic bipartite frame F, all edges in general position. Then every bond with at most three edges of the planarized drawing isolates one vertex, and the host obtained by replacing every crossing by a site is cyclically 4-edge-connected. (iv) Gluing two cubes onto a 4-cycle of a cyclically 4-edge-connected cubic graph, the cycle's edges deleted, keeps it cyclically 4-edge-connected.*

Proof outline. (i) If $|N(X)| \le |X| + 1$ for some X on one side with $2 \le |X| \le n - 2$, the cut around $X \cup N(X)$ has three edges and both sides contain cycles; the Hall-type characterization of braces [8] concludes. (ii) The rungs are the start rung, the two square edges of each junction and the rungs between junctions. (iii) A closed curve crossing at most three edges has at most one frame vertex inside (the frame is cyclically 4-edge-connected), and the one-vertex-per-face condition leaves room for at most one vertex of G; sites are ladders, so a site cut through costs two edges and its pieces with a cycle have four boundary edges. (iv) The twelve vertices of the junction are 2-edge-connected, their subsets with a cycle have at least four boundary edges, and the host minus the square has no bridge (`PLAIN-BRACE-20261031-001B`, Lemmas 1–5). □

Proof of Theorem 5. Take a hard plain source G (Theorem 3's proof), connected, and draw it with its vertices in distinct square faces of a prism $C_{2n} \times K_2$, $2n \ge |V(G)|$, drawn by Tutte's method [12]. Replace every crossing by a site with $k = 2t+1$ junctions. By Lemma 20 each instance is cyclically 4-edge-connected, hence a brace; it is Pfaffian by the certificate of Theorem 14 (every 4-cycle of the host is a face); it carries no sign. By Lemma 19 the counts for $t < \binom{m+3}{3}$ determine $\#\mathrm{col}(G \sqcup F) = \#\mathrm{col}(G)\#\mathrm{col}(F)$, and $\#\mathrm{col}(F) \neq 0$ because F has nice 4-cycles (Proposition 17) (`PLAIN-BRACE-20261031-001B`, Theorem 7). □

Corollary 21. *Among cubic triple systems whose incidence graphs are Pfaffian braces, the planar ones count in polynomial time (Valiant), the Fano plane is a single system, and the class is #P-hard. The Fano plane is the only one without twists.*

8 Remarks and open questions

The picture. On Pfaffian incidence graphs the count is governed by the sign class: the Pfaffian class gives one determinant per planar piece (Theorem 1), twists along a path of the trisum tree change nothing (Theorem 2), and branching twists make the count universal (Theorems 3–5). Two twisted sides at a square pass as a similitude when the third side is untwisted; three twisted sides make a soft crossing; a ladder of plain squares crosses exactly.

An Ising reading. By (1), $\#\mathrm{col}(B, \sigma)$ is an Ising partition function on B at the imaginary coupling with $x^2 = -\frac{1}{3}$, the signing giving the signs of the couplings; Pfaffian signings are its free-fermion sign patterns. We have not used this reading.

Open questions.

- *Few branched squares.* An even 4-ary signature with a nonzero entry at 0000 is a matchgate signature plus a multiple of the indicator of 1111. This suggests an algorithm exponential only in the number of squares with three twisted branches; we have not written it down.
- *Approximation.* Approximate counting on these classes is untouched; the degree-3 and edge-size-3 setting lies between the known algorithmic and hardness thresholds for hypergraph colourings.
- *Deficiency one.* Theorem 1 uses that H sits on Seymour's bound (n lines on n points). Systems with one line fewer are open.

9 Evidence and replay

Every theorem is proved in the repository, with a `check.py` that recomputes the cited facts and controls that must fail. Proof directories are `proofs/<ticket>/<ticket>-P001_<slug>/`.

Result	Proved in	Novelty, folklore risk
Lemma 6	`THREEFOLD-AMPLITUDE-20261030-001C`	signed form of Seymour's theorem; high
Theorem 8, Prop. 9	`SEYMOUR-COUNT-20261030-001A`, `SEYMOUR-COUNT-20261030-001B`	no statement found; moderate
Theorem 1	`SEYMOUR-COUNT-20261030-001B`	no statement found; moderate
Theorem 11, Theorem 2	`TWIST-COUNT-20261030-001A`	no statement found; low–moderate
Proposition 12	`TWIST-COUNT-20261030-001B`	exact computation; low
Lemma 13, Theorem 14	`SOFT-CROSS-20261031-001A`	standard method, new setting; moderate
Theorem 3	`SOFT-CROSS-20261031-001B`, `SOFT-CROSS-20261031-001C`	source known (Fan–Cai); moderate
Lemma 15, Theorem 4	`PLAIN-COUNT-20261031-001A`	standard method, new setting; moderate
Props. 16, 17	`PLAIN-COUNT-20261031-001B`	Prop. 17 likely known; high
Lemmas 18, 19	`PLAIN-BRACE-20261031-001A`	no statement found; low–moderate
Lemma 20, Theorem 5	`PLAIN-BRACE-20261031-001B`	no statement found; moderate

Computations. Among the checks: the similitude identity symbolically for all 32 pattern and sign choices; 13120 junctions of cube and 6-prism pages, none a crossover; 256 soft crossings among truncated-octahedron pages; $K_{3,3}$ recovered from Pfaffian instances under three signings $(54, 18, 18 = 9 \cdot (6, 2, 2))$; six formulas recovered through the source reduction on graphs up to 2710 vertices; two formulas recovered end to end through 44 certified Pfaffian instances; $K_{3,3}$ recovered from 100 counts of plain Pfaffian instances, the largest with 18504 vertices; 454 plain pages over 33 planar graphs, all twisted; woven brace instances of 214 to 1078 vertices, certified Pfaffian, with a site recovered exactly inside them.

Replay. `replay.py` reruns every cited check in a temporary copy and compares each fresh result with the frozen one, ignoring timings. Two checks depend on Python's per-run hash order: one labels the two weights of a child square in that order, the other draws random signings. There the replay accepts equal polynomials up to exchanging those weights, and equal numbers of signings of each kind with no failures; every other field must match exactly; `independent.py`, written separately without the repository's libraries, recomputes the soft crossing, the cube junction and its characteristic polynomial, the $K_{3,3}$ counts through plain braces, sign interpolation, the similitude

identity, and Theorem 1 on compositions of Fano planes; `check.py` ties every number quoted here to those runs. The review package lists the claims, the replay steps and questions for a reviewer.

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