Counting homomorphisms from knot groups to solvable groups can be #P-hard

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Abstract

For a finite group G and a link $L \subset S^3$, let $H_L(G)$ be the number of homomorphisms from $\pi_1(S^3 \setminus L)$ to G. Rowell conjectured in 2009 that $H_L(G)$ can be computed exactly in polynomial time from a diagram of L whenever G is solvable. Unless every function in #P is computable in polynomial time, this fails. On links it fails already for S_3 and the other dihedral groups of order $2n$ with n odd: tying torus knots into the components and cabling some of them turns the count into a Potts model, at the price of unboundedly many components. The main point of this note is that it fails on knots too, where metabelian targets are easy. Knot counting is #P-hard for S_4, the smallest group for which this can happen, and for its double covers $\mathrm{GL}(2,3)$ and $2O$. It is also #P-hard for groups of Fitting height two: for $\mathrm{UT}_3(\mathbb{F}_4) \rtimes C_3$, and for the Heisenberg groups $H(\mathbb{F}_p) \rtimes C_r$ (p odd) whose action has a "hard pair" of eigencharacters. For the second family we prove the converse: without a hard pair, the count is polynomial for knot groups and for closed 3-manifolds with bounded first homology. The S_4 proof rests on a "bicycle law" for transposition colourings of alternating diagrams with tripled crossings, which turns the count into a 12-state Potts-type model. The Heisenberg proofs read the count as the kernel size of a twisted triple cup product, which Borromean surgeries steer freely enough to encode Potts partition functions. For $\mathrm{UT}_3(\mathbb{F}_4) \rtimes C_3$ the reduction also preserves approximation, so approximating the count is #BIS-hard.

1 Results

For a finite group G and a link $L \subset S^3$, write

$$H_L(G) = |\operatorname{Hom}(\pi_1(S^3 \setminus L), G)|.$$

Up to normalization it is the link invariant of the modular category $\mathrm{Rep}(D(G))$. Rowell [14, §3.3.5] made the following conjecture.

> **Conjecture 3.2.** Let G be a finite group and L a link. (a) There exists an FPRAS for computing $H_L(G)$ for any group G. (b) Suppose G is solvable. Then there is a polynomial algorithm for exact computation of $H_L(G)$.

This note was prepared with the assistance of AI systems: Anthropic's Claude for the proofs, code and text, and OpenAI's Codex for two independent reviews of the underlying proofs. Every cited computation can be rerun from the public repository https://github.com/jscul/knotlab; Section 7 explains how.

Kuperberg and Samperton proved that for every nonabelian simple G and nontrivial conjugacy class C, counting the homomorphisms from knot groups to G that send the meridian into C is #P-complete [8]. They also wrote that they were "much less confident" in part (b) [7, §5.3]. Krovi and Russell had shown exact #P-hardness on links for the alternating groups A_m, $m \geq 5$, with meridians in a fixed class [6]. On the positive side, $H_K(G) = |G|$ for nilpotent G, and metabelian targets reduce to linear algebra over the Alexander module [8, §1]. Galindo [1] proved that $|\operatorname{Hom}(\pi_1 M, G)|$ is #P-hard for every nonabelian G when M ranges over closed 3-manifolds; his hard instances are graph manifolds whose first Betti number grows with the input. Samperton and Weiß [16] proved that class-two nilpotent targets are polynomial when $|H^2(M; Z(G))|$ is bounded, and named S_3 on 3-manifold groups as open. None of these addresses solvable targets on link or knot complements, and we found no statement in either direction.

On links, part (b) fails for a simple reason (Proposition 1): the components of a link may carry different conjugacy classes, and the sum over these choices encodes the Potts model, already for S_3. This uses unboundedly many components; for links with a bounded number of components every metabelian target gives a polynomial-time count. On knots, whose groups have abelianization $\mathbb{Z}$, nilpotent targets give $H_K(G) = |G|$ and metabelian targets reduce to linear algebra, so a counterexample must have derived length at least three. Our main results are such counterexamples.

All reductions below are polynomial-time Turing reductions, and links are given by diagrams. Recall that FP = #P would mean that every function in #P is computable in polynomial time; it implies P = NP.

Proposition 1 (links)**.** *For the dihedral group D_n of order $2n$ with $n \geq 3$ odd (so $D_3 = S_3$), computing $H_L(D_n)$ from a link diagram is #P-hard. For each fixed C and each metabelian G, $H_L(G)$ is computable in polynomial time on links with at most C components.*

Theorem 2 (S_4)**.** *For $G = S_4$, for its double covers $G = \mathrm{GL}(2,3)$ and $G = 2O$ (the binary octahedral group), and for the count of maps sending the meridian to a transposition, computing $H_K(G)$ from a knot diagram is #P-hard. Every group of order less than 24, and $\mathrm{SL}(2,3)$, gives a polynomial-time count, so S_4 is the smallest group with this property unless* FP = #P.

Let ω be a primitive cube root of unity in $\mathbb{F}_4$. For a field $\mathbb{F}_q$ let $H(\mathbb{F}_q)$ be the Heisenberg group of triples with $(x,y,z)(u,v,t) = (x+u,\ y+v,\ z+t+xv)$. For $r \mid q-1$ and $\alpha, \beta \in \mathbb{F}_q^\times$ with $\alpha^r = \beta^r = 1$, let

$$G_{\alpha,\beta} = H(\mathbb{F}_q) \rtimes C_r, \qquad h \cdot (x,y,z) = (\alpha x,\ \beta y,\ \alpha\beta z),$$

where h generates C_r. The group $\mathrm{UT}_3(\mathbb{F}_4) \rtimes C_3$ of order 192 is G_{ω^2,ω^2} over $\mathbb{F}_4$; it has trivial centre and Fitting height two. Write $S^3_r(K)$ for r-surgery on K; its first homology is $\mathbb{Z}/r$.

Theorem 3 ($\mathrm{UT}_3(\mathbb{F}_4) \rtimes C_3$)**.** *Let $G = \mathrm{UT}_3(\mathbb{F}_4) \rtimes C_3$. For every knot K,*

$$H_K(G) = |\operatorname{Hom}(\pi_1 S^3_3(K), G)| + 63.$$

Computing either number from a diagram of K is #P-hard.

Theorem 4 (Heisenberg dichotomy)**.** *Let p be an odd prime, $r \geq 2$ a divisor of $p-1$, and $G = G_{\alpha,\beta}$ over $\mathbb{F}_p$. Call (α,β) a* hard pair *if $\alpha \neq 1$, $\beta \neq 1$ and $\alpha\beta \neq 1$.*

1. *If (α,β) is not a hard pair, then $H_K(G)$ is computable in polynomial time from a knot diagram, and for every C the number $|\operatorname{Hom}(\pi_1 M, G)|$ is computable in polynomial time for closed orientable 3-manifolds M with $|H_1(M;\mathbb{Z})| \leq C$, given by triangulations.*

2. *If (α, β) is a hard pair, then computing $H_K(G)$, and computing $|\mathrm{Hom}(\pi_1 S^3_r(K), G)|$, from a diagram of K is* #P*-hard.*

For odd p the smallest groups with a hard pair are $G_{2,2}$ over $\mathbb{F}_5$ with $r = 4$ (order 500) and $G_{2,2}$ over $\mathbb{F}_7$ with $r = 3$ (order 1029); there is none with $r = 2$. Part (1) is a special case of a more general statement (Remark 14).

Corollary 5. *Unless* FP = #P*, Rowell's Conjecture 3.2(b) fails on links for the dihedral groups of Proposition 1, and already on knots for each of the groups shown* #P*-hard in Theorems 2–4.*

Proof. All these groups are solvable, and a polynomial algorithm for $H_L(G)$ would compute a #P-hard function in polynomial time. □

For $\mathrm{UT}_3(\mathbb{F}_4) \rtimes C_3$ the reduction also preserves approximation, so approximating $H_K(G)$ is #BIS-hard (Proposition 18); this bears on part (a).

The line, as tested. On knots, metabelian targets are easy, and the hard examples above have derived length three. Derived length three does not suffice: $\mathrm{SL}(2,3)$ and the groups $G_{\alpha,\beta}$ with $\alpha\beta = 1$, $\alpha \neq 1$, have derived length three and are polynomial. In the Heisenberg family the obstruction to lifting a map is a twisted *triple* product of first cohomology classes. Without a hard pair every unbounded part of it is linear or pairwise and Gauss sums count its zeros; with a hard pair it is a genuinely three-way form whose alternating part can be steered at will, and that freedom encodes the Potts model. The same picture separates $\mathrm{SL}(2,3)$ (quadratic obstruction, bounded target) from $\mathrm{UT}_3(\mathbb{F}_4) \rtimes C_3$ (cubic obstruction).

2 Preliminaries

Wirtinger colourings. A homomorphism from a link group to G is a colouring of the arcs of a diagram by elements of G such that at each crossing, with over-arc coloured g and incoming under-arc coloured x, the outgoing under-arc is coloured gxg^{-1} or $g^{-1}xg$ according to the sign of the crossing. The signs of individual crossings matter in general, but not when g is an involution, as for the transposition colourings of Section 4. Reversing the convention at every crossing at once replaces each colouring by its pointwise inverse, so counts over conjugacy classes closed under inversion do not depend on it. All meridians of a knot are conjugate and generate the group, so a homomorphism from a knot group sends every meridian into one conjugacy class.

Counting over a quotient. If $G = N \rtimes Q$ with N abelian, the homomorphisms $\pi \to G$ lying over a fixed $\psi\colon \pi \to Q$ are the 1-cocycles $Z^1(\pi; N_\psi)$, where N_ψ is N with π acting through ψ. For a central extension $1 \to A \to \tilde{G} \to G \to 1$, a map $\rho\colon \pi \to G$ lifts if and only if an obstruction class in $H^2(\pi; A)$ vanishes, and the lifts then form a torsor under $H^1(\pi; A)$; twisted versions of both statements hold for extensions by modules [13].

Potts and Tutte. For a multigraph Γ with n vertices, m edges and k components, the Potts partition function is

$$Z_\Gamma(Q, v) = \sum_{\sigma\colon V \to [Q]} \prod_{uv \in E} \big(1 + v[\sigma_u = \sigma_v]\big) = \sum_{S \subseteq E} Q^{k(S)} v^{|S|} = Q^k v^{n-k}\, T_\Gamma\Big(1 + \tfrac{Q}{v},\, 1 + v\Big),$$

where $k(S)$ counts the components of (V, S) and T_Γ is the Tutte polynomial. Evaluating T_Γ at a rational point (x, y) is #P-hard unless $(x-1)(y-1) = 1$ or (x, y) is one of eight special points [5]; on planar graphs it stays hard off the hyperbolas $(x-1)(y-1) \in \{1, 2\}$ and finitely many points [17]. In particular the chromatic polynomial $P(H, 12)$ of a planar graph is #P-hard to compute. The hardness proofs use interpolation, so they are Turing reductions.

3 Links: the dihedral groups

For a link L with a component K_i, write μ_i for its meridian and λ_i for its Seifert longitude, and sum over $\rho \in \mathrm{Hom}(\pi_1(S^3 \setminus L), G)$.

Lemma 6 (cables and local knots)**.** *1. Replacing K_i by k parallel copies, pushed off along the Seifert framing, gives $H = \sum_\rho |C_G(\rho(\lambda_i))|^{k-1}$.*

2. Tying a knot J into K_i gives $H = \sum_\rho N_J(\rho(\mu_i))$, where $N_J(g)$ counts the homomorphisms from the group of J that send its meridian to g. If G is metabelian, the new longitude has image $\rho(\lambda_i)$.

Proof. (1) The new exterior is the old one with the cable space ($D^2 \setminus k$ points) $\times S^1$ glued to the boundary torus of K_i, the boundary circle of the disk going to μ_i and the circle factor to λ_i. Its group is $F_k \times \mathbb{Z}$, so an extension of ρ chooses $a_1, \dots, a_k \in C_G(\rho(\lambda_i))$ with $a_1 \cdots a_k = \rho(\mu_i)$; since $\rho(\mu_i)$ commutes with $\rho(\lambda_i)$, there are $|C_G(\rho(\lambda_i))|^{k-1}$ choices. (2) The new exterior is the union of the two exteriors along a meridional annulus, so maps are pairs that agree on the meridian. The new longitude is $\lambda_i \lambda_J$, and λ_J lies in the second commutator subgroup, which a metabelian G kills. □

Proof of Proposition 1. Let $\Gamma = (V, E)$ be a multigraph with n vertices and m edges, and $c = n + m$. Let L_Γ be a link with an unknotted component U_v for each vertex and K_e for each edge, with $\mathrm{lk}(K_e, U_u) = 1$ and $\mathrm{lk}(K_e, U_v) = -1$ for $e = uv$ and all other linking numbers 0; the closure of the pure braid $\prod_{e=uv} A_{u,e} A_{v,e}^{-1}$ is one. Choose a prime $p \mid n$ and let J be the torus knot $T(2, p)$. A homomorphism from its group to D_n is determined by the meridian when the meridian is a rotation, and there are p of them when it is a fixed reflection, so N_J is 1 on rotations and p on reflections. Let $L(j)$ be L_Γ with j copies of J tied into every component and then every edge component replaced by two parallel copies. By Lemma 6,

$$H_{L(j)}(D_n) = \sum_{s=0}^{c} p^{js} B_s, \qquad B_s = \sum_{\rho:\ s \text{ reflection meridians}} \ \prod_{e \in E} |C(\rho(\lambda_e))|.$$

The $c+1$ values $j = 0, \dots, c$ determine B_0 through a Vandermonde system in the distinct nodes p^s. In B_0 every meridian goes to the cyclic subgroup C_n of rotations, so ρ factors through $H_1 = \mathbb{Z}^c$ and is any $x \in (\mathbb{Z}/n)^c$; then $\rho(\lambda_e) = x_u - x_v$, whose centralizer has order $2n$ if $x_u = x_v$ and n otherwise (n is odd). Hence

$$B_0 = n^m \sum_{x:\, V \to \mathbb{Z}/n} \ \prod_{uv \in E} n\big(1 + [x_u = x_v]\big) = n^{2m} Z_\Gamma(n, 1),$$

and $Z_\Gamma(n, 1) = n^k T_\Gamma(1+n, 2)$ lies on $(x-1)(y-1) = n \geq 3$, which is #P-hard [5] (also for planar Γ [17]). The link $L(j)$ has polynomial size. Using k-fold instead of 2-fold cables and interpolating in k recovers $Z_\Gamma(n, 2^{k-1} - 1)$ for several k, hence the number of proper n-colourings, without appeal to the Tutte classification.

For the second statement, the maps to the abelian quotient G/G' factor through $H_1 = \mathbb{Z}^c$, so there are at most $|G/G'|^C$ of them, and the lifts of each through the abelian G' solve a linear system (`BOUNDED-H1-20261025-001B`). □

Remark 7. More is true if one accepts Galindo's theorem [1], whose hard instances are closed manifolds given by planar surgery diagrams with framing 0. Replacing every component of such a diagram by k framed parallels, for polynomially many k, determines by interpolation the number of ρ that send every framed longitude into the centre of G, since only those reach the largest node $|G|^c$ (Lemma 6(1) with framings). For G with trivial centre this is the count of the surgered closed manifold, so $H_L(G)$ is #P-hard on links for every nonabelian G with trivial centre. We have not checked the proofs of [1]. For nilpotent targets the cables see framed longitudes only modulo the centre, and we do not know the complexity on links: for D_4 and Q_8 the count depends only on the linking numbers, as $4^c \sum_S 4^{-\operatorname{cutrank}(S)}$ over subsets S of the components, where $\operatorname{cutrank}(S)$ is the rank over $\mathbb{F}_2$ of the linking matrix with rows in S and columns outside S (verified on 60 random links each).

4 The symmetric group S_4

4.1 Reduction to transposition colourings

Let $T(K)$ and $R(K)$ count the homomorphisms $\pi_1(S^3 \setminus K) \to S_4$ that send the meridian to a transposition and to a 3-cycle respectively.

Lemma 8. *$H_K(S_4) = 4 + R(K) + 2T(K)$, and $R(K)$ is computable in polynomial time.*

Proof. Sort the maps by the class of the meridian's image. The identity gives one map. If the meridian goes to a double transposition, the image lies in the abelian normal subgroup V_4, so the map factors through $H_1 = \mathbb{Z}$: three maps. Livingston [9, Thm. 9] proved that over each Fox 3-colouring the lifts to S_4 with a transposition meridian and with a 4-cycle meridian are equally many, so the 4-cycle class contributes $T(K)$; for colourings with cyclic image both lifts lie in a dihedral group of order 8 and are counted directly. Finally a map with a 3-cycle meridian lands in $A_4 = V_4 \rtimes C_3$, which is metabelian, so $R(K)$ is a sum of two cocycle counts $|Z^1(\pi; \mathbb{F}_{4,\omega^{\pm 1}})|$, computed by linear algebra. □

The repository re-proves Livingston's identity (`CLOSED-H1-20261025-001B`). It remains to show that $T(K)$ is #P-hard.

4.2 Tripled alternating diagrams

Let $G = (V, E)$ be a connected plane multigraph with n vertices and m edges. Let $D(G)$ be the alternating link diagram whose Tait graph is G: the black regions are the vertices, and each edge is one crossing. Let $G^{(3)}$ repeat every edge three times side by side, so that $D(G^{(3)})$ has a twist of three crossings for each edge. For a spanning subgraph H write $c(H)$ for its number of components (isolated vertices included) and $B(H)$ for its bicycle space over $\mathbb{F}_2$: the edge sets that are both cycles and cocycles. Let $\mathrm{Col}(D)$ be the number of colourings of D by the six transpositions of S_4.

The transpositions of S_4 lie over the transpositions of $S_3 = S_4/V_4$, two over each, and the two commute. Label the transpositions of S_3 by $\mathbb{Z}/3$ so that conjugation reads $a \triangleright b = 2a - b$. A twist of k crossings acts on the colours at its top by the k-th power of the Hurwitz move $\sigma(x, y) = (x \triangleright y,\ x)$.

Lemma 9. *On pairs of transpositions of S_3, σ^3 is the identity. On pairs of transpositions of S_4 over colours a, b: if $a = b$ then $\sigma^3(x, y) = (y, x)$; if $a \neq b$ then σ^3 keeps the S_3-colours in place and changes each of the two lifts by an element of V_4 that depends only on (a, b).*

Proof. $\sigma^3(a, b) = (3b - 2a,\ b) = (a, b)$ modulo 3. Over equal colours the two transpositions commute, so $\sigma(x, y) = (y, x)$. The remaining case is a direct check on the 6×6 table of conjugates. □

Proposition 10 (the bicycle law)**.**

$$\operatorname{Col}\big(D(G^{(3)})\big) = \sum_{\varphi\colon V \to \mathbb{Z}/3} 2^{\,c(H_\varphi) + \dim B(H_\varphi)}, \qquad H_\varphi = \big(V, \{uv \in E : \varphi_u = \varphi_v\}\big).$$

Equivalently, with $\nu(t, \varphi, x) = (1-t, \varphi, 1-x)$ the fixed-point-free involution of $\mathbb{Z}/2 \times \mathbb{Z}/3 \times \mathbb{Z}/2 \cong \mathbb{Z}/12$, N its permutation matrix and J the all-ones matrix,

$$\operatorname{Col}\big(D(G^{(3)})\big) = 2^{-n} \sum_{z \in (\mathbb{Z}/12)^V} \prod_{uv \in E} M(z_u, z_v), \qquad M = J - 2N.$$

Proof. By Lemma 9 each twist passes S_3-colours along positions, as the smoothing that separates the two black regions would. That smoothing turns $D(G^{(3)})$ into one circle around each black region, so the S_3-colourings correspond to potentials $\varphi\colon V \to \mathbb{Z}/3$, and the twist of uv is monochromatic exactly when $\varphi_u = \varphi_v$.

Fix φ. A lift to S_4 chooses one of two transpositions on each arc. By Lemma 9 the choices follow the strands through monochromatic twists, which cross, and keep their positions, up to fixed elements of V_4, through the other twists, which is the smoothing that deletes the edge. So the lifts form an affine space whose dimension is the number of closed curves of the diagram of H_φ, each isolated vertex contributing its circle. The space is not empty: $S_3 = \mathrm{Stab}(4) \subset S_4$ is a complement of V_4 whose transpositions are transpositions of S_4. By Rosenstiehl and Read [15], [3, §17.3], a connected plane graph C has $1 + \dim B(C)$ left-right cycles, so the number of curves is $c(H_\varphi) + \dim B(H_\varphi)$.

For the second form, $c(H) + \dim B(H)$ is the nullity over $\mathbb{F}_2$ of the Laplacian $L_H\colon x \mapsto \delta x$ maps $\ker L_H$ onto $B(H)$ with the locally constant functions as kernel. The condition $L_H x = 0$ says that every vertex has an even number of H-edges whose ends carry different bits. Writing each parity condition as $\frac{1}{2}\sum_{t_v \in \mathbb{F}_2} (-1)^{t_v(\cdots)}$ gives the edge factor $(-1)^{(t_u + t_v)[\varphi_u = \varphi_v][x_u \neq x_v]} = 1 - 2[z_v = \nu(z_u)]$ with $z = (t, \varphi, x)$. □

Remark 11. If every edge of G is subdivided, summing out each middle vertex gives $\frac{1}{2}M^2 = 4J + 2I$ per edge, and $\mathrm{Col} = 2^{-n} 4^m Z_G(12, \frac{1}{2})$. The diagram is then a link with $1 + \beta_1(G)$ components, and one query to the colouring count evaluates the planar Tutte polynomial at $(25, \frac{3}{2})$, which is #P-hard. Knots need the gadgets below.

4.3 Knots

A two-terminal series-parallel network Γ, with every edge tripled, substituted for each edge of G, gives an interaction matrix M_Γ on $\mathbb{Z}/12$ after its internal vertices are summed (with a factor $\frac{1}{2}$ each). Series composition is the matrix product (with the factor $\frac{1}{2}$ for the middle vertex) and parallel composition the entrywise product. The span of J, I, N is closed under both, because $J^2 = 12J$, $JN = NJ = J$, $N^2 = I$, $J \circ X = X$, $N \circ N = N$ and $I \circ N = 0$. Writing $M_\Gamma = a\big((1+b)I + (1+g)N + (J - I - N)\big)$, parallel composition multiplies a, $1 + b$ and $1 + g$.

Lemma 12. *1. If τ counts spanning trees, t_Γ the spanning trees of Γ and f_Γ its spanning 2-forests separating the terminals, then $\tau(G[\Gamma]) = \tau(G)\, t_\Gamma^{n-1} f_\Gamma^{m-n+1}$.*
2. $D(K^{(3)})$ is a knot if and only if $\tau(K)$ is odd, for a connected plane multigraph K.

Proof. (1) A spanning tree of $G[\Gamma]$ meets each copy of Γ in a spanning tree or in a separating 2-forest, and the copies of the first kind form a spanning tree of G. (2) A twist with an odd number of crossings connects strands like one crossing, so $D(K^{(3)})$ and $D(K)$ have the same number of components, namely $1 + \dim B(K)$ by Rosenstiehl–Read. And $\dim B(K)$ is the $\mathbb{F}_2$-nullity of the reduced Laplacian, whose determinant is $\tau(K)$. □

The gadget $\Gamma_1 = S(e, P(e, S(e, e)))$ has $M_{\Gamma_1} = 17J + 8I - 2N$ (with the factors $\frac{1}{2}$ included), so $(1 + b, 1 + g) = (25/17, 15/17)$, $t = 3$ and $f = 5$. Its k-fold parallel power $\Gamma_1^{\|k}$ has $t_k = 3k \cdot 5^{k-1}$ and $f_k = 5^k$, both odd for odd k.

Proof of Theorem 2. Let H be a connected planar graph; $P(H, 12)$ is #P-hard. Double every edge outside a spanning tree T_0 to get $\tilde{H}$, with $\tilde{m}$ edges. Then $P(\tilde{H}, 12) = P(H, 12)$, and $\tau(\tilde{H}) = \sum_T 2^{|T \setminus T_0|}$ is odd. For odd $k = 1, 3, \ldots, 2K - 1$, with $K = (\tilde{m} + 1)(\tilde{m} + 2)/2$, the diagram $D\big((\tilde{H}[\Gamma_1^{\|k}])^{(3)}\big)$ is a knot by Lemma 12, with $12k\tilde{m}$ crossings, and by Proposition 10

$$T\big(D((\tilde{H}[\Gamma_1^{\|k}])^{(3)})\big) = 2^{-n}\, 17^{k\tilde{m}} \sum_{i,j} N_{ij}\, \beta_{ij}^k, \qquad \beta_{ij} = \left(\tfrac{25}{17}\right)^i \left(\tfrac{15}{17}\right)^j,$$

where N_{ij} counts the $z \in (\mathbb{Z}/12)^V$ with i edges having $z_u = z_v$ and j edges having $z_v = \nu(z_u)$. The $\beta_{ij} = 5^{2i+j} 3^j 17^{-i-j}$ are pairwise distinct, so the K counts determine every N_{ij} through a Vandermonde system in β_{ij}^2.

The reduction runs in polynomial time. It makes $K = O(\tilde{m}^2)$ queries, and each diagram has $12k\tilde{m} = O(\tilde{m}^3)$ crossings and as many arcs, so each count T is at most $6^{12k\tilde{m}}$ and has $O(\tilde{m}^3)$ bits. Each β_{ij} is a quotient of integers with $O(\tilde{m})$ bits and $k \le 2K - 1 = O(\tilde{m}^2)$, so every coefficient β_{ij}^k and every factor $2^{-n} 17^{k\tilde{m}}$ has $O(\tilde{m}^3)$ bits. The coefficient matrix of the system is a Vandermonde matrix in the β_{ij}^2 times the invertible diagonal matrix of the β_{ij}, and exact Gaussian elimination over $\mathbb{Q}$ solves it in time polynomial in these bit lengths. The reduction is a polynomial-time Turing reduction with K queries, not a parsimonious one.

Finally $P(H, 12) = \sum_j N_{0j}$: a colouring is proper exactly when no edge has equal ends, whatever happens on the ν-related edges. By Lemma 8, $T = (H_K(S_4) - 4 - R)/2$, so $H_K(S_4)$ is #P-hard as well.

For the double covers $\tilde{G} \in \{\mathrm{GL}(2, 3), 2O\}$, central extensions of S_4 by $\mathbb{Z}/2$: every map from a knot group to S_4 lifts, because H^2 of a knot exterior vanishes, and the lifts form a torsor under $H^1(\pi; \mathbb{F}_2) = \mathbb{F}_2$. So $H_K(\tilde{G}) = 2H_K(S_4)$.

For the smallest-group statement: every group of order less than 24 is metabelian (it has an abelian normal Sylow subgroup with abelian quotient, or is a 2-group of order at most 16, whose derived subgroup has order at most 4), and the only non-metabelian groups of order 24 are S_4 and $\mathrm{SL}(2, 3)$. Metabelian (indeed abelian-by-nilpotent) targets give polynomial counts on knots: maps to the nilpotent quotient are boundedly many because $H_1 = \mathbb{Z}$, and their lifts through the abelian kernel A are the solutions of a system of linear equations over A (`BOUNDED-H1-20261025-001B`). Writing $A = \mathbb{Z}/d_1 \times \cdots \times \mathbb{Z}/d_s$, these are integer congruences modulo the d_i, and the Smith normal form of the integer matrix that combines the coefficients with the moduli counts their solutions in polynomial time. This covers primary parts with several different exponents, which need not be

free over a single $\mathbb{Z}/p^a$. For $\mathrm{SL}(2,3)$, a central extension of A_4 by $\mathbb{Z}/2$, the argument for the double covers gives $H_K(\mathrm{SL}(2,3)) = 2H_K(A_4)$. □

5 Heisenberg groups twisted by a cyclic group

Throughout, $q = p$ is an odd prime with $r \mid p-1$, or $q = 4$ with $r = 3$, and $G = G_{\alpha,\beta}$. Let M be a closed oriented 3-manifold with $\pi = \pi_1 M$, and $\psi\colon \pi \to \mathbb{Z}/r$. For $\chi \in \mu_r(\mathbb{F}_q)$ let $\mathbb{F}_\chi$ be $\mathbb{F}_q$ with $g \in \pi$ acting by $\chi^{\psi(g)}$, and $V_\chi = H^1(M;\mathbb{F}_\chi)$. Put $\gamma = (\alpha\beta)^{-1}$.

5.1 The count is the kernel size of a triple product

Proposition 13 (`COPRIME-CLASS2-20261026-001B`, `CORRECTIONS-20261026-001A`)**.** *The homomorphisms $\pi \to G$ lying over ψ number*

$$|Z^1(\pi;\mathbb{F}_{\alpha\beta})| \cdot |B^1(\pi;\mathbb{F}_\alpha)| \cdot |B^1(\pi;\mathbb{F}_\beta)| \cdot \mathcal{N}_\psi, \qquad \mathcal{N}_\psi = \#\{(a,b) \in V_\alpha \times V_\beta :\ T(a,b,\cdot) = 0\},$$

where $T(a,b,c) = \langle a \smile b \smile c, [M]\rangle$ for $c \in V_\gamma$. For $\mathrm{UT}_3(\mathbb{F}_4) \rtimes C_3$ and $H_1(M) = \mathbb{Z}/3$ this gives $|\mathrm{Hom}(\pi, G)| = 1 + 128 \cdot 4^d \cdot \mathcal{N}_\psi$ with $d = \dim_{\mathbb{F}_4} V_{\omega^2}$.

Proof. The centre $A = \{(0,0,z)\}$ is invariant and $G/A = (\mathbb{F}_\alpha \oplus \mathbb{F}_\beta) \rtimes C_r$, because h preserves the section $(x,y) \mapsto (x,y,0)$. Maps to G/A over ψ are pairs of cocycles (c_x, c_y). The section has the bilinear cocycle $((x,y),(u,v)) \mapsto xv$, so the obstruction to lifting is $[c_x \smile c_y] \in H^2(\pi;\mathbb{F}_{\alpha\beta})$, and when it vanishes the lifts form a torsor under $Z^1(\pi;\mathbb{F}_{\alpha\beta})$. The class depends only on the cohomology classes $a = [c_x]$, $b = [c_y]$, which accounts for the factors $|B^1|$. Since $H^2(\pi) \to H^2(M)$ is injective and Poincaré duality pairs $H^2(M;\mathbb{F}_{\alpha\beta})$ perfectly with $H^1(M;\mathbb{F}_\gamma)$, the class vanishes exactly when $T(a,b,\cdot) = 0$. For the last formula, $\psi = 0$ gives only the trivial map (a nontrivial image in the 2-group $\mathrm{UT}_3(\mathbb{F}_4)$ would need 2-torsion in H_1), and the two surjections ψ are Galois conjugate and contribute $4 \cdot 4^d \cdot 4^2 \cdot \mathcal{N}_\psi$ each. □

Equivalently $\mathcal{N}_\psi = |V_\gamma|^{-1} \sum_{a,b,c} e(T(a,b,c))$ for a nontrivial additive character e of $\mathbb{F}_q$: the count is a cubic exponential sum.

5.2 Without a hard pair: polynomial

Proof of Theorem 4(1). There are at most $|H_1(M)|$ maps ψ, and for each k the characters $\alpha^k, \beta^k, (\alpha\beta)^k$ still include 1. The factors $|Z^1|$ and $|B^1|$ are linear algebra. If $\alpha^k = 1$ then $V_{\alpha^k} = H^1(M;\mathbb{F}_p)$ has at most $|H_1(M)|$ elements: enumerate a and count the kernel of the linear map $b \mapsto T(a,b,\cdot)$. The case $\beta^k = 1$ is symmetric. If $(\alpha\beta)^k = 1$, then V_{γ^k} is bounded with a basis $c_1, \dots, c_s$, and $\mathcal{N}$ counts the common zeros of s bilinear forms:

$$\mathcal{N} = p^{-s} \sum_{\theta \in \mathbb{F}_p^s} \sum_{a,b} e\Big(\textstyle\sum_i \theta_i T(a,b,c_i)\Big) = p^{-s} \sum_\theta p^{\dim V_{\alpha^k} + \dim V_{\beta^k} - \operatorname{rank} T(\cdot,\cdot,\sum \theta_i c_i)}.$$

All the inputs (cocycle spaces, cup products on a triangulation, the obstruction modulo coboundaries) are polynomial-time linear algebra. For a knot exterior X, which is aspherical, the count over ψ is the same with $\mathcal{N} = \#\{(a,b) : a \smile b = 0 \in H^2(X;\mathbb{F}_{(\alpha\beta)^k})\}$. Here ψ ranges over the r maps $\mathbb{Z} \to \mathbb{Z}/r$, $H^1(X;\mathbb{F}_p) = \mathbb{F}_p$ and $H^2(X;\mathbb{F}_p) = 0$, so each term is again an enumeration of a bounded space followed by linear algebra. □

Remark 14. The same argument proves more (COPRIME-CLASS2-20261026-001C). Let $G = N \rtimes C_r$ with N a p-group, $p \nmid r$, having a C_r-invariant central subgroup A with A and N/A elementary abelian and with N described by an equivariant bilinear cocycle (hypotheses (K)). Over a splitting field the commutator pairing has components $\kappa_{\chi,\chi'} \colon \bar{B}_\chi \times \bar{B}_{\chi'} \to \bar{A}_{\chi\chi'}$. If no component with χ, χ' and $\chi\chi'$ all nontrivial is nonzero, counting is polynomial with bounded first homology (for $p = 2$ the quadratic part is handled by Arf invariants). Whether the converse holds for all such N is open; Theorem 4 proves it for Heisenberg groups. Groups with higher-exponent layers are not covered.

5.3 Steering the triple product by Borromean surgery

A Y-graph (a clasper with three leaves) in M determines a Borromean surgery $M \rightsquigarrow M'$ [11, 4]. When the leaves ℓ_1, ℓ_2, ℓ_3 lie in $\ker \psi$, the cover and the cohomology are undisturbed.

Proposition 15 (COPRIME-CLASS2-20261026-001B(c), CORRECTIONS-20261026-001C)**.** *If the leaves lie in* $\ker \psi$*, then* $H_1(M') \cong H_1(M)$*,* ψ *extends, every* V_χ *is unchanged, and for characters* $\chi_1\chi_2\chi_3 = 1$

$$T'(a,b,c) - T(a,b,c) = \varepsilon \sum_{\sigma \in S_3} \operatorname{sgn}(\sigma)\, \lambda_{\sigma 1}(a)\, \lambda_{\sigma 2}(b)\, \lambda_{\sigma 3}(c),$$

where $\lambda_i(a)$ *evaluates a cocycle representing* a *on the based leaf* ℓ_i*, and* $\varepsilon = \pm 1$ *is a fixed orientation convention.*

Sketch. Since ψ vanishes on the boundary of a neighbourhood of the Y-graph, and H_1 of that boundary surjects onto H_1 of both the neighbourhood and the handlebody glued in, the r-fold cyclic cover is trivial over the surgery region and the twisted cocycles extend; this gives the first statements. Massuyeau's formula for the change of the triple cup product under a Y-surgery [10, Lemma 3.2], applied to the r lifts in the cover, adds r equal determinants, because the characters multiply to 1; the transfer divides by r, which is invertible in $\mathbb{F}_q$. Changing the basing path multiplies the three leaf functionals by $\chi_1(g), \chi_2(g), \chi_3(g)$, whose product is 1. □

This is the finite-cover, $\mathbb{F}_q$ analogue of Moussard's formula for null Y-graphs and equivariant triple intersections in the infinite cyclic cover [12, Prop. 2.5, Thm. 2.7]. Our leaves need only lie in $\ker \psi$.

Lemma 16 (pure leaves; COPRIME-CLASS2-20261026-001B(d))**.** *Let* $q = p$ *and suppose* ψ *is onto. For each* $\zeta \in \mu_r(\mathbb{F}_p)$ *and each functional* f *on* V_ζ *there is a loop in* $\ker \psi$ *whose functional is* f *on* V_ζ *and* 0 *on* V_χ *for every* $\chi \neq \zeta$*.*

Proof. H_1 of the cover with $\mathbb{F}_p$ coefficients surjects onto $\bigoplus_\chi V_\chi^*$, and every class comes from a loop in $\ker \psi$. Pick $t \in \pi$ with $\psi(t) = 1$ and apply the eigenprojector $P_\zeta = r^{-1} \sum_j \zeta^{-j} t^j$, realized by the product of the conjugates $t^j \ell t^{-j}$ raised to the powers $r^{-1}\zeta^{-j} \bmod p$. □

The hypothesis that ψ is onto is needed: for the lens space $L(7,1)$ with $p = 7$, $r = 3$ and $\psi = 0$ the three local systems coincide and no loop separates them. In the constructions below ψ sends the meridian to 1.

5.4 Gadgets

Fix a multigraph Γ with n vertices and m edges, and orient each edge $e = uv$. Use coordinates x_v ($v \in V$) and p_e, s_e, w_e ($e \in E$), and put $\ell_e = x_u - x_v$. For three functionals write $\lambda \wedge \mu \wedge \nu$ for the trilinear form $\det(\lambda, \mu, \nu)$, and for two $(\lambda \wedge \mu)(a, b) = \lambda(a)\mu(b) - \mu(a)\lambda(b)$. Proposition 15 can change the component (α, β, γ) only by forms that are alternating in every pair of slots with equal characters. The coincidences among α, β, γ give four patterns.

Lemma 17 (COPRIME-CLASS2-20261026-001D §1, CORRECTIONS-20261026-001D Lemma 6)**.**

Pattern	*Spaces*	*Form* T	$\mathcal{N}$
α, β, γ *distinct*	$X \oplus P \oplus S$ *in each slot*	$\sum_e \ell_e \wedge p_e \wedge s_e$	$q^{2m-n} Z_\Gamma(q^3, q^2 - 1)$
$\alpha = \beta = \gamma$	$V_\alpha = X \oplus P \oplus S$	$\sum_e \ell_e \wedge p_e \wedge s_e$	$q^{2m-n} Z_\Gamma(q^3, q^2 - 1)$
$\alpha = \beta \neq \gamma$	$V_\alpha = X \oplus P$, $V_\gamma = W$	$\sum_e (\ell_e \wedge p_e) \otimes w_e$	$q^m Z_\Gamma(q^2, q - 1)$
$\gamma = \alpha \neq \beta$	$V_\alpha = X \oplus P$, $V_\beta = S$	$\sum_e s_e(b)\,(\ell_e \wedge p_e)(a, c)$	$q^{m-n} Z_\Gamma(q^2, q - 1)$

For $q = 4$, where a hard pair forces $\alpha = \beta = \gamma$, the second row holds with $\ell_e = x_u + x_v$. The case $\gamma = \beta \neq \alpha$ is symmetric to the last.

Proof. Write $\mathcal{N} = |V_\gamma|^{-1} \sum_{a,b,c} e(T(a, b, c))$ and sum the edge coordinates first; the sums factor over the edges. For the three-way form fix the vertex parts and let $u_e = (\ell_e(a), \ell_e(b), \ell_e(c)) \in \mathbb{F}_q^3$. Then $\sum_{P \in \mathbb{F}_q^{3\times 2}} e(\det(u_e \mid P))$ is q^6 if $u_e = 0$ and q^4 otherwise: move u_e to the first basis vector by GL_3, after which $\det = p_2 s_3 - p_3 s_2$ and each product sum gives q. Colouring each vertex by $(a_v, b_v, c_v) \in \mathbb{F}_q^3$, the total is $q^{4m} \sum_\sigma \prod_e (1 + (q^2 - 1)[\sigma_u = \sigma_v])$, and dividing by $|V| = q^{n+2m}$ gives the first two rows. For $\alpha = \beta \neq \gamma$ the condition at e is $\ell_e(a)p_e(b) - p_e(a)\ell_e(b) = 0$, which has q^2 solutions $(p_e(a), p_e(b))$ if $\ell_e(a) = \ell_e(b) = 0$ and q otherwise. For $\gamma = \alpha \neq \beta$ the conditions are $s_e(b)\ell_e(a) = 0$ for each edge and $\sum_{e \ni w} \pm s_e(b)p_e(a) = 0$ for each vertex; with $S = \operatorname{supp} s$ the second system has $q^{m-n+k(S)}$ solutions $p(a)$, and the first forces S-edges to be monochromatic, so $\mathcal{N} = q^{m-n} \sum_S (q - 1)^{|S|} q^{2k(S)}$. □

All four Tutte points $(1 + q^3/(q^2 - 1), q^2)$ and $(1 + q^2/(q - 1), q)$ lie on $(x - 1)(y - 1) = q^3$ or q^2, so evaluating the Potts functions above is #P-hard [5].

5.5 Realizing the gadgets by knots

Base knots. For odd p and $\zeta \in \mathbb{F}_p^\times \setminus \{1\}$, the twist knot K_m with Alexander polynomial $mt^2 - (2m + 1)t + m$ and $m \equiv \zeta(\zeta - 1)^{-2} \pmod p$ has ζ and ζ^{-1} as roots modulo p. Its Alexander module is cyclic, so for $S_r^3(K_m)$ with ψ the meridian character, V_ζ and $V_{\zeta^{-1}}$ are one-dimensional and all other V_χ vanish. (For example 6_1 has roots 2 and $1/2$ for every $p \geq 5$.) For $q = 4$ use the figure-eight knot, with V_ω and V_{ω^2} one-dimensional over $\mathbb{F}_4$. In a connected sum the twisted cohomology and T split over the summands, because the splitting annulus has no twisted cohomology. Every component of T vanishes on the base: a nonzero component would need three characters with product 1 from one summand's $\{\zeta, \zeta^{-1}\}$, forcing $\zeta^3 = 1$ on a one-dimensional space, where an alternating form is zero; for 4_1 over $\mathbb{F}_4$ the form is zero by direct computation. With a hard pair no two of α, β, γ are inverse to each other, so the summands for different slots do not interfere.

Construction (COPRIME-CLASS2-20261026-001D §3, CORRECTIONS-20261026-001D §3). Given Γ and the pattern:

1. Take one base summand for each coordinate of each *distinct* character space of the pattern, drawn in disjoint balls B_i joined by bands, with a base point and a reference path into each ball.
2. In each B_i pick arcs a_i, b_i on which the summand's nonzero class takes different values (constant cochains are coboundaries), and let γ_i be the loop around a_i positively and b_i negatively. It bounds a disk in B_i meeting the knot twice, and its functional is nonzero on the summand's block. For odd p apply Lemma 16 inside B_i to get a *pure* leaf supported on the slot's eigenspace.
3. Realize ℓ_e as a band sum of the leaves of the summands of u and v (the second reversed), choosing the band so that the closed loop through it has $\psi \equiv 0$; otherwise unbalanced gains change the count.
4. Expand the gadget into decomposable terms and give each term one Y-graph whose leaves are parallel push-offs of the leaves it names, with edges routed so that every leaf, read from the Y-graph's vertex, has its reference value, and every leaf has the framing of its disk.
5. Each term has a leaf carrying a coordinate that occurs in no other factor of that term (p_e, s_e or w_e). Choose its cap disk disjoint from all other Y-graphs; parallel caps suffice when another term uses the same coordinate. Forgetting the knot, that leaf bounds a disk, so the Y-surgery gives back S^3 [4, Props. 3.3–3.4], and the diffeomorphism is the identity near a meridian. Hence the result is $S^3_r(K_\Gamma)$ for a knot K_Γ, with the same framing.

The Y-graphs and their leaf disks are pairwise disjoint, so the surgeries are independent tangle replacements, and K_Γ has $O(m(n+m))$ crossings. By Proposition 15, applied once per Y-graph, the component (α, β, γ) of T on $S^3_r(K_\Gamma)$ is the gadget form.

The other characters. By Proposition 13, $|\operatorname{Hom}(\pi_1 S^3_r(K_\Gamma), G)| = 1 + \sum_{k=1}^{r-1} c_k \mathcal{N}_k$, where $c_k > 0$ depends only on dimensions and $\mathcal{N}_k$ reads the component $(\alpha^k, \beta^k, \gamma^k)$. With pure leaves only components that are permutations of (α, β, γ) change, and each such $\mathcal{N}_k$ equals the gadget count; every other component keeps its base value 0, so its $\mathcal{N}_k = |V_{\alpha^k}||V_{\beta^k}|$. Over $\mathbb{F}_4$ the two surjections are Galois conjugate and contribute equally. In all cases the oracle answer is an affine function of the gadget count with known coefficients.

5.6 Knot groups

Proof of Theorem 3. Let $N = \mathrm{UT}_3(\mathbb{F}_4)$ and $\rho\colon \pi_1(S^3 \setminus K) \to G$, with meridian μ and preferred longitude λ. If $\rho(\mu) \in N$, the image is a finite 2-group normally generated by one element, hence cyclic (its abelianization is cyclic), so there are $|N| = 64$ such maps; for them $\rho(\lambda) = 1$, and $\rho(\mu)^3 = 1$ forces $\rho(\mu) = 1$, so only the trivial one satisfies $\mu^3\lambda = 1$. Otherwise λ lies in the second commutator subgroup, so $\rho(\lambda) \in G'' = Z(N)$; it commutes with $\rho(\mu) \in Nh^{\pm 1}$, which acts on $Z(N)$ by $\omega^{\pm 1}$ without fixed points, so $\rho(\lambda) = 1$. Every element of $Nh^{\pm 1}$ has order 3, so $\rho(\mu)^3\rho(\lambda) = 1$ holds automatically. Hence $H_K(G) = |\operatorname{Hom}(\pi_1 S^3_3(K), G)| + 64 - 1$. Hardness: given Γ, build K_Γ with base $\#^{n+2m} 4_1$, query once, and solve $1 + 128 \cdot 4^{n+2m} \cdot 4^{2m-n} Z_\Gamma(64, 15)$ for $Z_\Gamma(64, 15)$, a #P-hard Tutte evaluation at $(79/15, 16)$. □

Proof of Theorem 4(2). For the surgeries this is the construction above with one oracle call per Tutte evaluation. For the knot exterior X of K_Γ: if a character χ is nontrivial on the meridian, the boundary torus and the surgery solid torus have no χ-twisted cohomology, so $H^i(X; \mathbb{F}_\chi) = H^i(S^3_r(K_\Gamma); \mathbb{F}_\chi)$ for $i = 1, 2$, compatibly with cup products. Hence a term of the count whose three characters are all nontrivial on the meridian is the same for X as for $S^3_r(K_\Gamma)$. Terms in which one of the three characters is trivial fall under the algorithm of part (1). So $H_{K_\Gamma}(G)$ is a computable quantity plus a known positive multiple of the gadget count. □

6 Remarks and open questions

What is tested. For links, $B_0 = 9^m Z_\Gamma(3,1)$ is recovered from directly computed counts $H_{L(j)}(S_3)$ on five graphs (the largest link has 2088 crossings and 12 components), the proper 3-colourings of a path and a triangle (12 and 6) are recovered from link counts, and the same holds for D_5 and D_7 on three graphs; the counting method is checked against brute force on random braids, and Lemma 6 on separate cases. The identities of Sections 4 and 5 are checked by brute force on small cases (Section 7). The steering law is checked in all four patterns against direct computation of the obstruction on braid closures with Borromean insertions. Complete graph-to-knot instances were computed for the triangle and the digon over $\mathbb{F}_4$ ($\mathcal{N} = 45162496$ and 326656), for patterns two, three and four over $\mathbb{F}_5$ and $\mathbb{F}_7$ ($\mathcal{N} = 3240625$, 46101601 and 25925), with leaf configurations found by search and equal to the gadget up to a change of basis. The band-and-projector recipe of the construction has not been run end to end, and pattern one (which needs $r \geq 6$) has steering checks but no complete instance.

Approximate counting. Part (a) of Rowell's conjecture asks for an FPRAS for every G. All our reductions land on ferromagnetic Potts points ($v > 0$), and for one group the reduction preserves approximation.

Proposition 18. *For $G = \mathrm{UT}_3(\mathbb{F}_4) \rtimes C_3$, approximating $H_K(G)$ is #BIS-hard under approximation-preserving reductions. So part (a) of Rowell's conjecture for this G would give an FPRAS for #BIS.*

Proof. By the proofs of Theorem 3 and Proposition 13, the knot K_Γ built from a multigraph Γ with m edges has $H_{K_\Gamma}(G) = 64 + 128 \cdot 4^{4m} Z_\Gamma(64, 15)$: over $\mathbb{F}_4$ both nontrivial characters are steered, so no unsteered term remains. One query with relative error δ, minus the known constant 64 and divided by $128 \cdot 4^{4m}$, approximates $Z_\Gamma(64,15)$ with relative error at most 2δ. Approximating the ferromagnetic Potts partition function $Z_\Gamma(Q, v)$ with $Q > 2$ and $v > 0$ is #BIS-hard on general graphs [2, Thm. 1]. □

This is evidence against part (a), not a refutation. The other reductions do not preserve approximation as written. For the Heisenberg groups with pure leaves, the component read by ψ^{-1} is never a permutation of (α, β, γ), so it keeps its maximal value and dominates the count. For S_4 the knot reduction interpolates, and the link reduction of Remark 11 lands on the planar ferromagnetic Potts model, for which we know no #BIS-hardness. The interactions induced by kernel counts are positive semidefinite, which suggests that this route gives at most #BIS-type hardness of approximation.

Open questions.

1. The converse of Remark 14 for all coprime class-two groups with a hard pair, and groups with higher-exponent layers.
2. Extension fields other than $\mathbb{F}_4$; for $q = 2^k$ in patterns 2 and 4 a twisted Bockstein term can appear.
3. Approximation for the Heisenberg groups and for S_4; upper bounds ($H_K(G) \leq_{\mathrm{AP}}$ #BIS) for any solvable G.
4. Closed 3-manifolds with bounded first homology for $\mathrm{GL}(2,3)$ and $2O$ (for S_4 itself the repository proves hardness with $H_1 = \mathbb{Z}/2$, `CLOSED-H1-20261025-001C`).

7 Evidence and replay

The proofs, each with a standard-library check, are in the public repository https://github.com/jscul/knotlab under `proofs/<ticket>/`. The replay driver `replay.py` in `proofs/ROWELL-NOTE-20261026-001/` copies them to a temporary directory, reruns every check there, and compares each fresh result with the frozen one; the frozen files are not touched.

Claim	Proof and check
Links: S_3 and D_n (n odd) hard; cables and local knots	`notes/knot-group-counting/links/` (`run_all.py`)
Bicycle law, Potts$(12, \frac{1}{2})$, knot gadgets, $T(K)$ #P-hard	`BOUNDED-H1-20261025-001A`
Abelian-by-nilpotent targets polynomial on knots	`BOUNDED-H1-20261025-001B`
Livingston's identity re-proved	`CLOSED-H1-20261025-001B`
$H_K(S_4) = 4 + R + 2T$; double covers	`SL23-BOUNDED-20261025-001B`
Count as a triple-product kernel ($\mathbb{F}_4$)	`CORRECTIONS-20261026-001A`
Steering ($\mathbb{F}_4$)	`CORRECTIONS-20261026-001C`
$\mathrm{UT}_3(\mathbb{F}_4) \rtimes C_3$ hard; knot groups (+63)	`CORRECTIONS-20261026-001D`
Count, steering and pure leaves (odd p)	`COPRIME-CLASS2-20261026-001B`
No hard pair: polynomial; Remark 14	`COPRIME-CLASS2-20261026-001C`
Hard pair: gadgets, construction, knot groups	`COPRIME-CLASS2-20261026-001D`

An independent review with fresh code (`COPRIME-CLASS2-20261026-001A`) reproduced every number of the $\mathbb{F}_4$ case and 894 further steering configurations; a second review replayed the four gadget patterns and the +63 identity with separate code. Both were performed by AI systems and are no substitute for expert review.

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